Pricing Distressed CDOs with Stochastic Recovery*

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Abstract

In this article, a framework for the joint modelling of default and recovery risk in a portfolio of credit risky assets is presented. The model especially accounts for the correlation of defaults on the one hand and correlation of default rates and recovery rates on the other hand. Nested Archimedean copulas are used to model different dependence structures. For the recovery rates a very flexible continuous distribution with bounded support is applied, which allows for an efficient sampling of the loss process. Due to the relaxation of the constant 40% recovery assumption and the negative correlation of default rates and recovery rates, the model is especially suited for distressed market situations and pricing of super senior tranches. A calibration to CDO tranche spreads of the European iTraxx portfolio is performed to demonstrate the fitting capability of the model. Applications to delta hedging as well as base correlations are presented.

Keywords: CDO, Nested Archimedean Copula, Stochastic Recovery

JEL classification: G13

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1 Introduction

Standard copula models for the pricing of collateralized debt obligations (CDOs) assume a constant recovery rate of 40%. While this assumption might work quite well in normal market situations, in distressed markets, as observed since the 2^{nd} half of 2007, this assumption is not justified anymore: First, standard copula models, like the Gaussian copula model introduced by Li (2000), often show a bad performance in times of high tranche spreads. Second, in 2008 it was temporarily not possible to calibrate the standard Gaussian base correlation model to the complete set of CDX and iTraxx tranche quotes (see e.g. Krekel (2008)). And finally, non-standardized super senior tranches ($60\% - 100\%$) have a fair spread of zero in standard market models with a 40% recovery rate while being traded on the market with a positive spread of up to 25 bps during distressed market situations.
Nevertheless, only very few CDO models with stochastic recovery exist. The first one was introduced in Andersen and Sidenius (2004). In this article, an extension of the Gaussian copula model is presented by assuming a stochastic recovery related to the systematic factor driving the default events, explicitly allowing for a negative correlation between recovery rates and default rates. To be more precise, the recovery rate of an obligor in case of a default in this model is given by an application of the normal cumulative distribution function (cdf) on a normally distributed random variable which is correlated with the default triggering variable through a common systematic factor. In a numerical examination the authors noted that the base correlation skew effect of random recovery is quite minor and hence the random recovery approach was not further investigated. Due to the credit market crisis, recently some articles on using stochastic recovery rates in CDO pricing have been published. Krekel (2008) uses a discrete stochastic recovery rate in a Gaussian base correlation setting to overcome the problem that super senior tranches in a standard Gaussian base correlation model have zero fair spread. In this model the discrete recovery rates are defined as constants on buckets of the default triggering factors, i.e. the recovery rate is a step function of the default triggering variable. In the empirical part a recovery rate distribution with only four possible realizations (60%, 40%, 20%, and 0%) is used. Amraoui and Hitier (2008) extend the approach of Krekel (2008) by modelling the recovery rate as a deterministic function of the systematic risk factor of the default triggering variable. Ech-Chatbi (2008) uses a multiple default approach (similar to Section 6.1.3 in Schönbucher (2003)), where the recovery is lowered by a random factor each time a default event occurs. Hence, the recovery rate process is some geometric compound Poisson process where the current recovery rate is multiplied by a random variable, e.g. beta distributed or log-gamma distributed, each time a default event occurs. One feature that all these models have in common is that they rely on the assumption of a Gaussian copula, which might not be appropriate,

especially in distressed market situations, as Gaussian copulas don't support tail dependences.
The aim of this article is the joint modelling of default and recovery risk in a portfolio of credit risky assets, especially accounting for the correlation of defaults on the one hand and correlation of default rates and recovery rates on the other hand. Nested Archimedean copulas as proposed in Hofert and Scherer (2009) are used to model different dependence structures. However, this concept is not applied to model different default dependences for firms in the same sector and firms from different sectors as in Hofert and Scherer (2009), but rather to model dependences between default triggers (inner dependence) as well as between default triggers and loss triggers (outer dependence). Furthermore, a very flexible continuous recovery-rate distribution with bounded support on $[0, 1]$ is chosen, which allows for an efficient sampling of the loss process. This is especially important as in most cases the loss process distribution will not be given in closed form.
The organization of the article will be as follows: The modelling framework is introduced in Section 2. Section 3 contains the payment streams of all considered credit derivatives. Section 4 discusses exchangeable and nested Archimedean copulas as a concept of dependence modelling, while in Section 5 a concrete choice for the recovery rate distribution is given. A calibration procedure and a numerical example with market data are presented in Section 6. Section 7 shows how to extend the concept of base correlations to the presented modelling framework. Section 8 concludes.

2 Modelling Framework

In the following a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ is assumed, where $\mathbb{Q}$ denotes some given pricing measure. Furthermore, let $(\mathcal{F}_t)_{t\in[0,T]}$ with $\mathcal{F} = \bigcup_{t\in[0,T]} \mathcal{F}_t \subset \mathcal{G}$, $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$, denote the background filtration representing the information about the financial market except for information on the occurrence or non-occurrence of default events. Consider a portfolio of I credit risky assets with payment streams depending both on the default status and on the recovery rate in case of default.
The default times τ_i, $i = 1, \ldots, I$, are assumed to follow a default intensity model with $\lambda_i(t)$ denoting the $\mathcal{F}_t$-measurable default intensity of firm i at time t. The univariate survival probabilities in this model are given by

$$p_i(t) := \mathbb{Q}(\tau_i > t) = \mathbb{E}\left[e^{-\int_0^t \lambda_i(s)ds} \right] \tag{1}$$

and the default probabilities are denoted by $\bar{p}_i(t) = 1 - p_i(t)$. According to p.183 of Bielecki and Rutkowski (2004) or p.122 of Schönbucher (2003), τ_i can be constructed in the canonical way as follows: Let U_i^D denote a random variable uniformly distributed on $[0,1]$ and independent of $\mathcal{F}$, then

$$\tau_i \stackrel{d}{=} \inf\left\{t > 0 : e^{-\int_0^t \lambda_i(s)ds} \leq U_i^D\right\}. \tag{2}$$

The recovery rates R_i in case of a default event are given by $1-LGD_i$, where the losses given default, LGD_i, are assumed to be identically distributed according to a distribution function F with support $[0,1]$; if no default has taken place, R_i is set equal to 1. This choice is arbitrary as the recovery rate is only of importance for defaulted assets. Hence, the recovery rate of firm i is given by

$$R_i = \begin{cases} 1 - LGD_i & \text{if } \tau_i \leq T \\ 1 & \text{else} \end{cases}.$$

Assuming U_i^L to be uniformly distributed on $[0,1]$ and independent of $\mathcal{F}$, one can set $LGD_i = F^{\leftarrow}(U_i^L)$ and $R_i = h(U_i^L, \tau_i)$ with $F^{\leftarrow} := \inf\{z : F(z) \geq x\}$ denoting the pseudo-inverse of F and

$$h(x,\tau) = \begin{cases} 1 - F^{\leftarrow}(x) & \text{if } \tau \leq T \\ 1 & \text{else} \end{cases}.$$

Setting e.g. $F^{\leftarrow}(x) \equiv 0.6$ for $x \in (0,1)$ leads to the standard case of a constant recovery of 40%.

To introduce dependence between different firms on the one hand and between recovery rates and default rates on the other hand, a certain dependence structure between the trigger variables U_i^D, U_i^L, $i = 1,\ldots,I$, is assumed. The joint distribution of the vector $\left(U_1^D,\ldots,U_I^D,U_1^L,\ldots,U_I^L\right)$ is assumed to be given by some copula C, i.e. $\left(U_1^D,\ldots,U_I^D,U_1^L,\ldots,U_I^L\right) \sim C$. E.g. in the standard Gauss copula model with constant recovery as introduced in Li (2000) C would simply be a Gaussian copula with correlation matrix

$$\Sigma = \begin{pmatrix} 1 & & \rho & & & \\ & \ddots & & & 0 & \\ \rho & & 1 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix},$$

correlation coefficient $\rho \geq 0$, and $F^{\leftarrow}(x) \equiv 0.6$ for $x \in (0,1)$.

To allow for different dependence hierarchies, C is chosen to belong to the class of nested Archimedean copulas in what follows. The concept of pricing CDO tranches using exchangeable Archimedean copulas was introduced in Schönbucher and Schubert (2001). Hofert and Scherer (2009) extended the approach by using nested Archimedean copulas to introduce different default correlations for different industry sectors. In the following, this concept will be applied to allow for dependence between default rates and recovery rates.

The most important quantity for pricing portfolio-credit derivatives as well as for risk management purposes is the portfolio-loss process $L(t)$ which can be easily derived once the default process, loss given default, and nominal of each asset in the portfolio are known. Unfortunately, the distribution of the loss process is generally not known in closed form. In some cases, e.g. homogeneous portfolio, exchangeable Archimedean copula, and constant recovery (see Proposition 10.7 in Schönbucher (2003)), the portfolio-loss distribution can be approximated by a conditional independence approach. Nevertheless, as long as the occurring processes can be sampled efficiently, it is possible to price portfolio-credit derivatives via a Monte Carlo approach.

3 Portfolio CDS and CDO tranches

In the following, the payment streams and pricing formulas of portfolio CDS and CDOs are presented. To accomplish this, consider a portfolio of I obligors, where each obligor contributes the same amount to the notional. For ease of notation a notional of 1 is assumed for the whole portfolio in what follows. The time to maturity is denoted by T. For the pricing of CDS and CDOs the payment streams of two different legs, the premium and the default leg, have to be considered. Premium payments are made at predefined dates given by the payment schedule $\mathcal{T} = \{0 < t_1 < \ldots < t_n = T\}$. Note that a default event can happen at any point in time in the interval $[0, T]$. To simplify computations, all default payments between two premium payment dates are defered to the next scheduled payment date. Furthermore, to account for accrued interest, defaults are assumed to happen in the middle of two scheduled payment dates, i.e. at $(t_{k-1} + t_k)/2$ for a default event in $[t_{k-1}, t_k)$ with $k = 1, \ldots, n$ and $t_0 = 0$. Both portfolio CDS and CDO tranches might be seen as credit derivatives with the (relative) portfolio-loss process $L(t)$ given by

$$L(t) = \frac{1}{I} \sum_{i=1}^{I} 1_{\{\tau_i \leq t\}} LGD_i, \ t \in [0, T]$$

as underlying. The basic idea of a CDO is to pool the risky assets and resell them in slices, the so-called CDO tranches. The loss affecting tranche $j \in \{1, \ldots, J\}$ with lower and upper attachement points l_j and u_j, $0 = l_1 < u_1 = l_2 < \ldots < u_{J-1} = l_J < u_J \leq 1$, is given by

$$L_j(t) = \min\{\max\{0, L(t) - l_j\}, u_j - l_j\}, \ t \in [0, T].$$

With each default the nominal of the portfolio of obligors is reduced by $1/I$, i.e. the remaining nominal of the portfolio CDS is

$$N(t) = 1 - \frac{1}{I} \sum_{i=1}^{I} 1_{\{\tau_i \leq t\}}, \ t \in [0, T].$$

For tranche $j \in \{1, \dots, J\}$ the remaining nominal is given by

$$N_j(t) = u_j - l_j - L_j(t),\ t \in [0, T].$$

With s_T^{pCDS} denoting the annualized portfolio-CDS spread, $d(t)$ the discount factor for the period $[0, t]$ at time 0, and $\Delta t_k = (t_k - t_{k-1}),\ k = 1, \dots, n,$ the time between two subsequent scheduled payment dates, the expected discounted premium and default leg of a portfolio CDS are given by

$$EDPL_T(s_T^{pCDS}) = \mathbb{E}\left[\sum_{k=1}^{n} d(t_k) s_T^{pCDS} \Delta t_k \left(N(t_k) + \left(N(t_{k-1}) - N(t_k)\right)/2\right)\right] \tag{3}$$

and

$$EDDL_T = \mathbb{E}\left[\sum_{k=1}^{n} d(t_k)\left(L(t_k) - L(t_{k-1})\right)\right] \tag{4}$$

respectively. For the j-th tranche of a CDO, the expected discounted premium and default leg are given by

$$\begin{aligned} & EDPL_{T,j}(s_{T,j}^{CDO}) \\ & = \mathbb{E}\left[\sum_{k=1}^{n} d(t_k) s_{T,j}^{CDO} \Delta t_k \left(N_j(t_k) + \left(N_j(t_{k-1}) - N_j(t_k)\right)/2\right)\right] \end{aligned} \tag{5}$$

with $s_{T,j}^{CDO}$ denoting the annualized spread of tranche j and

$$EDDL_{T,j} = \mathbb{E}\left[\sum_{k=1}^{n} d(t_k)\left(L_j(t_k) - L_j(t_{k-1})\right)\right] \tag{6}$$

respectively. It is market standard to quote the equity tranche, i.e. the most subordinated tranche, with an upfront payment (percentage of the nominal) and a fixed running spread of 500 bps, i.e.

$$\begin{aligned} EDPL_{T,1}(s_{T,1}^{CDO}) &= s_{T,1}^{CDO}(u^1 - l^1) \\ &\quad + \mathbb{E}\left[\sum_{k=1}^{n} d(t_k) 0.05 \Delta t_k \left(N_1(t_k) + \left(N_1(t_{k-1}) - N_1(t_k)\right)/2\right)\right], \end{aligned}$$

where $s_{T,1}^{CDO}$ denotes the upfront payment.
The fair spreads of both portfolio CDS and CDO tranches can now be computed by equating the expected discounted premium and default leg and solving for the spread.

4 Archimedean Copulas

In this section, the basic concepts of exchangeable and nested Archimedean copulas, which will be used to create dependence between defaults of different firms as well as between default rates and recovery rates, are repeated. A general introduction to copulas is, e.g., given by Joe (1997) or Nelsen (1998). One of the distinct properties of Archimedean copulas is that they are fully specified by some generator function. Furthermore, Archimedean copulas are flexible to capture various dependence structures, e.g. tail dependence. Applications of Archimedean copulas in financial modelling can e.g. be found in Schönbucher (2003) or Cherubini *et al.* (2004). In the last years, nested Archimedean copulas, which extend the concept of exchangeable Archimedean copulas by allowing for some asymmetries, have become quite popular in financial market applications (see e.g. Savu and Trede (2006) or Hofert and Scherer (2009)). Exchangeable Archimedean copulas are defined as follows.

Definition 1. *An I-dimensional exchangeable (i.e. distributionally invariant under permutations) Archimedean copula is given by*

$$C(u) = C(u_1, \ldots, u_I; \varphi_0) = \varphi_0^{-1}\left[\varphi_0(u_1) + \ldots + \varphi_0(u_I)\right], \ u \in [0,1]^I,$$

where the generator $\varphi_0 : [0,1] \mapsto [0,\infty]$ is a continuous and strictly decreasing function, which satisfies $\varphi_0(1) = 0$.

According to Kimberling (1974), C defines a proper copula in each dimension I if and only if the inverse of the generator φ_0^{-1} is a completely monotonic function, i.e.

$$(-1)^k \frac{d^k}{dt^k}\varphi_0^{-1}(t) \geq 0, \ k \in \mathbb{N}, \ t > 0.$$

Following Bernstein's Theorem (see, e.g. p. 439 of Feller (1971)) the class of completely monotonic functions φ_0^{-1} on $[0,\infty)$ with $\varphi_0^{-1}(0) = 1$ coincides with the class of Laplace-Stieltjes transforms of distribution functions G on $[0,\infty)$, i.e. $\varphi_0^{-1}(t) = \int_0^\infty e^{-tx} dG(x)$ for $t \geq 0$. Using this representation, Marshall and Olkin (1988) presented an efficient algorithm for sampling from multi-dimensional exchangeable Archimedean copulas under the assumption that G is known.

Algorithm 1.

1. *Sample from a random variable $V \sim G$.*
2. *Sample from i.i.d. random variables $X_i \sim Unif[0,1]$, $i \in \{1, \ldots, I\}$.*
3. *Return the random vector $(U_1, \ldots, U_I)$ with $U_i = \varphi_0^{-1}(-\log(X_i)/V)$.*

A more flexible multivariate Archimedean copula can be constructed by the nesting of generators. These copulas allow for different degrees of positive dependence in different margins.

Definition 2. *An I-dimensional partially nested Archimedean copula is given by*

$$\begin{aligned} C(u) &= C\left(C\left(u_{11},\ldots,u_{1d_1};\varphi_1\right),\ldots,C\left(u_{H1},\ldots,u_{Hd_H};\varphi_H\right);\varphi_0\right) \\ &= \varphi_0^{-1}\left[\varphi_0\left(\varphi_1^{-1}\left[\varphi_1(u_{11})+\ldots+\varphi_1(u_{1d_1})\right]\right)+\ldots\right. \\ &\quad \left.+\varphi_0\left(\varphi_H^{-1}\left[\varphi_H(u_{H1})+\ldots+\varphi_H(u_{Hd_H})\right]\right)\right] \\ &= \varphi_0^{-1}\left[\sum_{h=1}^{H}\varphi_0\left(\varphi_h^{-1}\left[\sum_{k=1}^{d_h}\varphi_h\left(u_{hk}\right)\right]\right)\right] \end{aligned} \tag{7}$$

with $u_{hk}\in[0,1]$, $h\in\{1,\ldots,H\}$, $k\in\{1,\ldots,d_h\}$, *and* $\sum_{h=1}^{H} d_h = I$.

Here, d_h denotes the dimension of the h-th subgroup. In the application to pricing CDOs presented in Section 6 there will be $H=2$ subgroups representing default and loss triggers. Each subgroup contains $d_1=d_2=125$ members representing a portfolio of 125 obligors according to the iTraxx conventions.

The function C from Equation (7) is a copula if all $\varphi_0\circ\varphi_h^{-1}$ have completely monotonic derivatives (see McNeil (2008)). In the special case that the generators belong to the same one-parameter family of Archimedean copulas, i.e. $\varphi_h=\varphi_0(\cdot;\theta_h)$, it is often sufficient to claim that $\theta_0\le\theta_h$, $h\in\{1,\ldots,H\}$, with θ_h denoting the parameter corresponding to φ_h, $h\in\{0,\ldots,H\}$ (see Hofert (2008)). This assumption will be used throughout the remainder of the article. Note that if the generators belong to different families the above condition rarely holds and is difficult to verify. An alternative approach to find compatible generators and to sample the resulting copula is presented in Hering *et al.* (2009).

Based on Algorithm 1, McNeil (2008) suggested an algorithm for sampling from partially nested Archimedean copulas. This algorithm applies Algorithm 1 iteratively by sampling from distribution functions associated with Laplace-Stieltjes transforms which are the inverses of the generators denoted by $\varphi_0(\cdot;\theta_h)$. Here, G again denotes the distribution function with Laplace-Stieltjes transform $\varphi_0^{-1}(\cdot;\theta_0)$.

Algorithm 2.

1. *Sample from a random variable* $V\sim G$.
2. *For* $h\in\{1,\ldots,H\}$ *sample from a random vector*

$$(X_{h1},\ldots,X_{hd_h})\sim C\left(u_{h1},\ldots,u_{hd_h};\varphi_0(\cdot;\theta_h)\right)$$

according to Algorithm 1.

3. Return the random vector $(U_{11}, \ldots, U_{Hd_H})$ *with*

$$U_{hk} = \varphi_0^{-1}\left(-\log(X_{hk})/V\right), \; h \in \{1, \ldots, H\}, \; k \in \{1, \ldots, d_h\}.$$

As mentioned above, in most (non-trivial) cases, there will be no closed-form approximation for the loss distribution. Nevertheless, if efficient sampling strategies from the recovery rate distribution are known, Monte Carlo techniques can be applied to price derivatives of the loss process.

5 Recovery-rate Distribution

As long as there are no liquidly traded credit derivatives on pure default-event risk, e.g. digital default swaps, or on pure recovery risk, e.g. recovery swaps, it is not possible to infer default intensities from credit derivatives without making an assumption on recovery rates. Therefore, while choosing the recovery-rate distribution, one has to ensure that the assumptions made in the pricing of correlation-insensitive credit derivatives used for the determination of default intensities are not violated in the model for correlation-sensitive products. For this, the expected recovery rate is assumed to equal the constant recovery rate, R_{const}, used for bootstrapping the default intensities from portfolio-CDS quotes, i.e.

$$\mathbb{E}\left[R_i | \tau_i \leq T\right] = R_{const}. \tag{8}$$

Using standard assumptions this means that the recovery distribution conditioned on default has to have an expectation of 40%. Otherwise the model is not consistent with the portfolio-CDS pricing.
The beta distribution, which is often used for recovery rates / loss given defaults (see e.g. Section 6.1 of Schönbucher (2003), Schuermann (2004), or Gupton and Stein (2005)), is numerically too expensive for Monte Carlo pricing (for the cdf and inverse cdf the incomplete beta function has to be evaluated numerically). Therefore, the Kumaraswamy distribution (see Kumaraswamy (1980)), which is a special case of McDonald's generalized beta distribution (see e.g. McDonald (1984) or Johnson *et al.* (1995)), is more suitable for Monte Carlo applications and hence chosen as marginal distribution for the loss given default. Its density is given by

$$f_{Kum}(x) = abx^{a-1}\left(1 - x^a\right)^{b-1},$$

where $0 \leq x \leq 1$ and $a, b > 0$. The Kumaraswamy distribution has similar properties as the beta distribution: it is a continuous distribution with bounded support showing a high degree of flexibility by supporting bimodal as well as unimodal and skewed, J-shaped, U-shaped, and uniform densities (see Figure 1(a)). In contrast to the beta distribution it has a closed-form cdf and inverse cdf given by

$$F_{Kum}(x) = 1 - (1 - x^a)^b$$

and

$$F_{Kum}^{-1}(x) = \left(1 - (1-x)^{1/b}\right)^{1/a}. \tag{9}$$

Other possible continuous distributions with bounded support for the loss given default that show a similar degree of flexibility would be the two-sided power distribution as proposed by Kotz and van Dorp (2004) or the Vasicek distribution (see Vasicek (1991)). The two-sided power distribution has similar properties as the Kumaraswamy distribution but a peak at its mode (antimode), whereas the Vasicek distribution has the drawback that for its cdf and inverse cdf the normal cdf and inverse cdf have to be evaluated numerically.
To ensure that condition (8) is fulfilled the parameters a and b have to be chosen such that the expectation of the recovery rate conditioned on default is equal to 40%, i.e. the expectation of the loss given default conditioned on default is equal to 60%. Therefore, set (see McDonald (1984))

$$\mathbb{E}\left[LGD_i|\tau_i \leq T\right] = bB\left(1 + \frac{1}{a}, b\right) = 0.6, \tag{10}$$

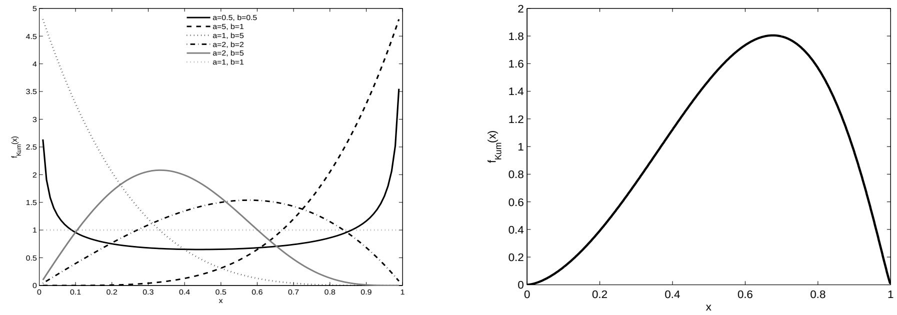
where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ denotes the incomplete beta function. The remaining degree of freedom can be resolved e.g. by choosing the second parameter such that the recovery distribution matches the empirically observed standard deviation (e.g. 20% as in Altman and Kishore (1996)), i.e.

$$\sigma_{LGD} := \sqrt{\mathbb{V}ar\left[LGD_i|\tau_i \leq T\right]} = \sqrt{bB\left(1 + \frac{2}{a}, b\right) - b^2 B\left(1 + \frac{1}{a}, b\right)^2} = 0.2. \tag{11}$$

This leads to a Kumaraswamy distribution with parameters $a = 2.65$ and $b = 2.13$ for the loss given default with density as given in Figure 1(b). By choosing other parameter constellations that fulfil Equation (10), one can vary the shape of the distribution. Another possibility would be to calibrate the second parameter to the market quotes of CDO tranches.
By choosing other parameter constellations that fulfil Equation (10), one can vary the shape of the distribution. Another possibility would be to calibrate the second parameter to the market quotes of CDO tranches.

6 Model Calibration and Empirical Results

In this section, a procedure to calibrate the model to market data is presented and a numerical example of the model's fitting capability using iTraxx Europe data is given. First of all, portfolio CDS and CDO tranches are priced using the results from Sections 2 - 5.



(a) Different parameter constellations. (b) With parameters $a = 2.65$ and $b = 2.13$.

Figure 1: Densities of Kumaraswamy distribution.

6.1 Pricing and Calibration Algorithm

Assuming deterministic discount factors, Equations (3) and (4) only require the computation of the expected portfolio loss and the expected remaining nominal at time t, which are given by

$$\mathbb{E}\left[L(t)\right] = \frac{1 - R_{const}}{I} \sum_{i=1}^{I} \bar{p}_i(t) \text{ and } \mathbb{E}\left[N(t)\right] = \frac{1}{I} \sum_{i=1}^{I} p_i(t).$$

with R_{const} denoting the constant recovery rate used for bootstrapping default intensities. Therefore, the model spread of a portfolio CDS can be obtained by equating Equations (3) and (4) and solving for s_T^{pCDS}, i.e.

$$s_T^{pCDS,model} = \frac{EDDL_T}{EDPL_T(1)}. \tag{12}$$

Here, the linearity of the premium and the default leg in the loss process is used. Unfortunately, premium and default leg for a CDO tranche are non-linear functions in the loss process. Therefore, a straightforward calculation of the expected discounted premium and default leg is not possible. Nevertheless, the following algorithm shows how to price CDO tranches in the modelling framework presented above via Monte Carlo simulation.

Algorithm 3. *Pricing CDO tranches via Monte Carlo simulation.*

1. *Let T denote the maturity, $\mathcal{T} = \{0 < t_1 < \ldots < t_n = T\}$ the payment schedule, $\lambda_i(t)$ the default intensity of firm i, I the number of firms, R_{const} the constant recovery rate used for bootstrapping default intensities from portfolio-CDS spreads, and $d(t_k)$ the discount factor at time t_k with $k = 0, \ldots, n$ and $t_0 = 0$. Choose the number of simulation runs M, the attachement and detachement point l_j and u_j of tranche j, and a copula C with parameter vector θ.*

2. *Monte Carlo simulation:*

   (a) *Sample* $\lambda_i^{(m)}(t_k)$ *and compute* $\Lambda_i^{(m)}(t_k) = e^{-\int_0^{t_k} \lambda_i^{(m)}(s)ds}$ *for* $k = 0, \dots, n$, $i = 1, \dots, I$, *and* $m = 1, \dots, M$.

   (b) *Sample* $U = (U_{i,m})_{i=1,\dots,2I, m=1,\dots,M} \in [0,1]^{2I \times M}$, *where each column* $U_{\cdot,m}$ *of* U *is a sample of the chosen copula* C.

   (c) *Compute the default times* $\tau_i^{(m)}$ *for each firm* $i \in \{1, \dots, I\}$ *and Monte Carlo run* $m \in \{1, \dots, M\}$ *via Equation (2) using* $\Lambda_i^{(m)}(t_k)$ *and* $U_{i,m}$ *from steps 2(a) and (b).*

   (d) *Compute the loss given default for each defaulted firm via*

$$LGD_i^{(m)}|_{\tau_i^{(m)} \le T} = F_{Kum}^{-1}\left(\widetilde{F}_{U_{I+i,\cdot|\tau_i^{(\cdot)} \le T}}\left(U_{I+i,m|\tau_i^{(m)} \le T}\right)\right) \tag{13}$$

   with $\widetilde{F}_{U_{I+i,\cdot|\tau_i^{(\cdot)} \le T}}$ *denoting the empirical cdf of* $U_{I+i,\cdot|\tau_i^{(\cdot)} \le T}$ *and* F_{Kum}^{-1} *the inverse cdf of the Kumaraswamy distribution from Equation (9). Here,* $U_{I+i,\cdot|\tau_i^{(\cdot)} \le T}$ *denotes the subsample of all* $U_{I+i,m}$ *for which* $\tau_i^{(m)} \le T$.

   (e) *Define the recovery rate of firm* i *as*

$$R_i^{(m)} = \begin{cases} 1 - LGD_i^{(m)}|_{\tau_i^{(m)} \le T} & \text{if } \tau_i^{(m)} \le T \\ 1 & \text{else} \end{cases}. \tag{14}$$

   (f) *Compute the loss process* $L^{(m)}(t_k)$ *for* $t_k \in \mathcal{T}$ *according to*

$$L^{(m)}(t_k) = \frac{1}{I}\sum_{i=1}^{I} 1_{\{\tau_i^{(m)} \le t_k\}}(1 - R_i^{(m)}).$$

   (g) *Compute for each tranche* $j \in \{1, \dots, J\}$ *and each* $t_k \in \mathcal{T}$ *the loss affecting tranche* j, $L_j^{(m)}(t_k)$, *and the remaining nominal in tranche* j, $N_j^{(m)}(t_k)$, *via*

$$L_j^{(m)}(t_k) = \min\{\max\{0, L^{(m)}(t_k) - l_j\}, u_j - l_j\}$$

   and

$$N_j^{(m)}(t_k) = u_j - l_j - L_j^{(m)}(t_k).$$

3. *Compute for each tranche* $j \in \{1, \dots, J\}$ *and each Monte Carlo run the discounted premium and default legs and estimate their expectations* $EDPL_{T,j}(1)$ *and* $EDDL_{T,j}$ *from Equations (5) and (6) by their*

sample means $\overline{EDPL}_{T,j}(1)$ *and* $\overline{EDDL}_{T,j}$.
Determine the fair spread of tranche $j \in \{2, \dots, J\}$ *via*

$$\widehat{s}_{T,j}^{CDO,model} = \frac{\overline{EDDL}_{T,j}}{\overline{EDPL}_{T,j}(1)}$$

and $\widehat{s}_{T,1}^{CDO,model}$ *via*

$$\begin{aligned}\widehat{s}_{T,1}^{CDO,model} &= \frac{1}{u_1 - l_1} \Big[\, \overline{EDPL}_{T,1}(1) \\ &\quad - \mathbb{E}\left[\sum_{k=1}^{n} d(t_k) 0.05 \Delta t_k \left(N_1(t_k) + \left(N_1(t_{k-1}) - N_1(t_k) \right) / 2 \right) \right] \Big].\end{aligned}$$

The motivation to choose the loss given default for each defaulted firm according to Equation (13) is the following: For each firm i only the subset of loss triggers is considered for which a default has occurred. This subset is denoted by $U_{I+i,\cdot|\tau_i^{(\cdot)} \le T}$. As this is no longer a sample of a uniformly distributed random variable, it is transformed by applying its empirical cdf $\widetilde{F}_{U_{I+i,\cdot|\tau_i^{(\cdot)} \le T}}$. This set of transformed realizations is uniformly distributed. By applying the inverse cdf of the Kumaraswamy distribution, the set of loss given defaults $LGD_i^{(\cdot)}|_{\tau_i^{(\cdot)} \le T}$ has the desired distributional properties. Furthermore, note that due to the positively quadrant dependence property (see e.g. Section 5.2.1 of Nelsen (1998)) of the considered nested copulas, loss given defaults of companies defaulting earlier in time are on average larger than those of obligors defaulting later. This effect might to some extent be justified by the empirical observation that the time to default of an obligor is positively (negatively) correlated with its recovery rate (loss given default), see e.g. Emery *et al.* (2004).
With these pricing routines, the model can now be calibrated to market quotes of portfolio-CDS spreads $s_T^{pCDS,market}$ and CDO tranche spreads $s_{T,j}^{CDO,market}$, $j = 1, \dots, J$. Since default intensities are specified independently from the dependence structure in the modelling framework presented above, the default-intensity process can be fitted to portfolio-CDS quotes in a first step and then the copula parameter vector θ can be fitted to the dependence structure induced by market quotes of CDO tranches. The former can be done by adjusting the default-intensity parameters such that the model spread in Equation (12) equals the market spread. For the latter the difference in model and market CDO tranche spreads has to be minimized over the Copula parameter vector θ. Since the equity tranche is quoted in terms of an upfront payment and not in terms of a running spread as the other tranches, care has to be taken when comparing pricing errors of different tranches. Accounting for this and the fact that the highest spread is paid for the equity tranche, the model is calibrated by minimizing the

deviation of market and model spreads for tranches $j = 2, \ldots, J$ provided that the model upfront payment matches the market upfront payment up to a certain accuracy ϵ reflecting bid-ask spreads, e.g. $\epsilon = 10^{-4}$ (see e.g. Hofert and Scherer (2009)). Hence, the following optimization problem has to be solved:

$$\begin{aligned} \min_{\theta} \; D_2 \;&:=\; \sum_{j=2}^{J} \left| s_{T,j}^{CDO,model} - s_{T,j}^{CDO,market} \right| \\ \text{s.t. } D_1 \;&:=\; \left| s_{T,1}^{CDO,model} - s_{T,1}^{CDO,market} \right| \leq \epsilon, \end{aligned} \tag{15}$$

where the minimization is taken over the involved copula parameter vector θ.

6.2 Calibration Results

The following numerical example shows that already in a very simple form the model fits market data quite well and leads to significantly smaller pricing errors compared to a standard Gaussian copula model. The following simplifying assumptions are made:
As the main focus of this article is on the modelling of recovery rates, the dependence between defaults, and the dependence between default rates and recovery rates, a homogeneous portfolio with constant default intensities $\lambda_i(t) = \lambda$ is assumed. Of course this framework could be generalized to the case of time-varying or stochastic default intensities at the cost of a numerically more expensive pricing algorithm.
Furthermore, three different parametric copula families with parameter vector θ will be compared for modelling dependence between default and loss triggers: Gaussian (Ga), Gumbel (Gu), and outer power Clayton (opC) copula. Each copula is tested in its exchangeable form with constant recoveries and in its nested form with stochastic recoveries. For the latter this means in terms of Section 4, $H = 2$, $d_1 = d_2 = 125$ (iTraxx standard), $\theta = (\theta_0, \theta_1, \theta_2)$, where the copula generators are given by $\varphi_0(\cdot;\theta_0)$, $\varphi_0(\cdot;\theta_1)$, and $\varphi_0(\cdot;\theta_2)$. Gumbel and outer power Clayton (with additional parameter θ_c) have been chosen from the family of Archimedean copulas, since they have shown consistently the best fitting results in the case of constant recovery rates (see Hofert and Scherer (2009)). These two copulas are compared to the Gaussian copula as a benchmark which is still some kind of market standard. Table 1 contains the parameter ranges, generator functions and inverses, and lower and upper tail dependence parameters of the considered copulas (see, e.g. Hofert and Scherer (2009)).
To keep the dimension of the optimization problem small, the inner copula parameters, describing the dependence among default triggers and the dependence among loss triggers, are assumed to be identical, i.e. $\theta_{in} := \theta_1 = \theta_2$.

Table 1: Copula properties.

Family	θ	$\varphi_0(t)$	$\varphi_0^{-1}(t)$	λ_L	λ_U
Gauss	$[-1,1]$	-	-	0	0
Gumbel	$[1,\infty)$	$(-\log(t))^\theta$	$\exp\left(-t^{\frac{1}{\theta}}\right)$	0	$2-2^{\frac{1}{\theta}}$
opC	$[1,\infty)$	$(t^{-\theta_c}-1)^\theta$	$(1+t^{\frac{1}{\theta}})^{-\frac{1}{\theta_c}}$	$2^{-\frac{1}{\theta\theta_c}}$	$2-2^{\frac{1}{\theta}}$

Different correlations for different industry sectors are also not under consideration here. This could be done as an extension of the model by adding another hierarchy level to the model. Furthermore, the additional parameter of the outer power Clayton copula is set to $\theta_c = 0.1$ (see Hofert and Scherer (2009)). Hence, the calibration to the market quotes of CDO tranches reduces to a two-dimensional optimization problem over the parameter vector $\theta = (\theta_{out}, \theta_{in})$ with $\theta_{out} := \theta_0$ denoting the outer copula parameter. This calibration is done in a two-step procedure similar to Algorithm 6 in Hofert and Scherer (2009).

The investigated dataset consists of portfolio-CDS and CDO market quotes of the 8^{th} and 9^{th} iTraxx Europe series between February and July 2008. Portfolio-CDS spreads and spreads of the first five CDO tranches with detachment and attachment points given by $[0\%, 3\%]$, $[3\%, 6\%]$, $[6\%, 9\%]$, $[9\%, 12\%]$, and $[12\%, 22\%]$ with a maturity of 5 years are used. According to the iTraxx Europe convention a quarter-yearly payment schedule $\mathcal{T}$ is used, i.e. $n = 4T$ and $T = 5$. The portfolio consists of $I = 125$ companies.

Five different, randomly picked trading days were chosen to test the modelling approach (2008-02-22, 2008-03-31, 2008-05-02, 2008-06-27, and 2008-07-25). In the following only the results for 2008-05-02 are discussed in detail. As all other results are quite similar they are deferred to Appendix A. The portfolio-CDS spread at this date was 63.74 bps, which leads to a constant default intensity of $\lambda = 0.0106$. This corresponds to a five-year default probability of $\bar{p}(5) = 5.17\%$.

First of all, the model is calibrated to the market spreads. Table 2 contains the calibrated parameters $\theta = (\theta_{out}, \theta_{in})$ and the default correlation given by (see e.g. Section 10.8 of Schönbucher (2003))

$$\rho(t) = \frac{\varphi_{in}^{-1}\left[2\varphi_{in}(p(t))\right] - p^2(t)}{p(t)(1-p(t))},$$

with $\varphi_{in}(t) = \varphi_0(t; \theta_{in})$. In addition to that, the average (simulated) correlation between default rates and recovery rates $\widehat{\rho}^{D,R}(5)$ is reported. Of course, in case of deterministic recovery rates $\widehat{\rho}^{D,R}(5)$ is not well-defined.

Table 3 contains the market and model upfront payment (in %) for tranche 1, the market and model spreads (in bps) for tranches 2 - 5, and the pricing errors D_2 (see Equation (15)) and D_2^{rel}, where D_2^{rel} is D_2 divided by the

Table 2: Calibrated parameters and correlations for 2008-05-02.

Model	θ_{in}	θ_{out}	$\rho(5)$	$\widehat{\rho}^{D,R}(5)$
Ga det	0.34		12.93%	-
Ga sto	0.28	0.24	8.40%	-43.35%
Gu det	1.26		26.06%	-
Gu sto	1.19	1.11	20.68%	-29.03%
opC det	1.25		25.77%	-
opC sto	1.16	1.16	18.49%	-48.39%

sum of market spreads of tranches 2 - 5, i.e.

$$D_2^{rel} := \frac{\sum_{j=2}^{J} \left| s_{T,j}^{CDO,model} - s_{T,j}^{CDO,market} \right|}{\sum_{j=2}^{J} s_{T,j}^{CDO,market}}.$$

The corresponding asymptotic two-sided 95% confidence intervalls are given in Table 9 in Appendix A.

Table 3: Market and model spreads for 2008-05-02.

Model	$0-3\%$	$3-6\%$	$6-9\%$	$9-12\%$	$12-22\%$	D_2	D_2^{rel}
Ga det	29.59%	496.48	250.50	142.08	53.12	411.87	77.67%
Ga sto	29.68%	488.42	241.94	137.92	52.90	390.86	73.70%
Gu det	29.63%	278.80	150.14	104.52	65.00	68.15	12.85%
Gu sto	29.60%	256.67	138.64	97.59	61.27	37.18	7.01%
opC det	29.69%	285.15	152.87	107.70	64.86	80.26	15.13%
opC sto	29.66%	270.51	142.35	97.73	59.98	48.43	9.13%
Market	29.65%	259.09	122.55	101.83	46.84	-	-

By introducing additional dependence between default rates and recovery rates, the default correlation gets lower and the pricing error smaller across all models.[1] For the Gaussian model this gain in performance is the smallest. Both the Gumbel and the outer power Clayton model can generate higher default correlations than the Gaussian model, which leads to consistently lower pricing errors. A possible explanation for this observation is the positive upper tail dependence of both Gumbel and outer power Clayton copula compared to the zero tail dependence of the Gaussian copula. Nevertheless, the outer power Clayton model needs a much higher correlation of default rates and recovery rates (with only a slightly lower default correlation) as the Gumbel model to generate similar fitting results. The absolute (relative) error for tranches 2 - 5 in the Gumbel model is 37.18 bps

[1] Note that the computational effort to achieve this is reasonable, e.g. implemented in Matlab R2009a the computing times for the spreads in the models with stochastic recovery presented in Table 3 (based on 100,000 Monte Carlo runs) on a Intel(R) Core(TM)2 Duo processor (2.4 GHz) with 2.96 GB RAM are between 70 and 80 seconds.

(7.01%). This result seems to be quite good especially as the sum of bid-ask spreads of tranches 2 -5 on this date is already 34.56 bps (6.16%). Since the Gumbel copula shows consistently better calibration results compared to the outer power Clayton copula, the following analyses will be restricted to a comparison of Gaussian and Gumbel copula, both with and without stochastic recoveries.

6.3 Parameter Sensitivity

In the next step, the sensitivity of the tranche spreads w.r.t. the copula parameter vector θ is investigated, while all other parameters are set to the calibrated values from Subsection 6.2.[2] For the Gumbel copula model it can be seen from Figure 2, that for tranches 1 and 2 the spread or upfront payment respectively decreases with increasing dependence parameters, whereas for tranches 4 and 5 the opposite holds true, i.e. holders of tranche 1 and 2 are short correlation, holders of tranche 4 and 5 are long correlation. In tranche 3 things are a bit different. While the spread decreases with increasing outer dependence parameter θ_{out}, it first increases with increasing inner dependence parameter θ_{in} until it reaches its maximum and decreases afterwards. Across all tranches the inner dependence parameter which drives the default correlation as well as the correlation of loss triggers has the higher impact on the tranche spreads. The outer dependence parameter which drives the dependence of default rates and recovery rates has its highest impact on the equity tranche. This effect can also be observed in the Gaussian model, but with a lower impact compared to the Gumbel case.

For the Gauss copula model it can be seen from Figure 3, that for tranches 1 and 2 the spread or upfront payment respectively again decreases with increasing dependence parameters. For tranches 3 - 5 the spread increases with increasing θ_{in} until it reaches its maximum and decreases afterwards.

Figure 4 shows the default correlation $\rho(5)$ and the average correlation between default rates and recovery rates $\widehat{\rho}^{D,R}(5)$ in the models with Gumbel and Gauss copula. As expected the default correlation increases with increasing inner dependence parameter θ_{in}, while the main driver of $\widehat{\rho}^{D,R}(5)$ is the outer dependence parameter θ_{out}.

Besides on the copula parameter vector $\theta = (\theta_{out}, \theta_{in})$, the tranche spreads also depend on the parameters of the recovery distribution a and b. So far they have been chosen such that Equations (10) and (11) are fulfilled. While one parameter has to be determined by Equation (10) to ensure consistency with portfolio-CDS pricing, the other parameter can be chosen arbitrarily. Figure 5 shows the tranche spreads for the Gauss and the Gumbel copula model and the loss given default standard deviation σ_{LGD} from Equation (11) for different values of a while all other parameters are set according

[2] All sensitivities are computed numerically using difference quotients.

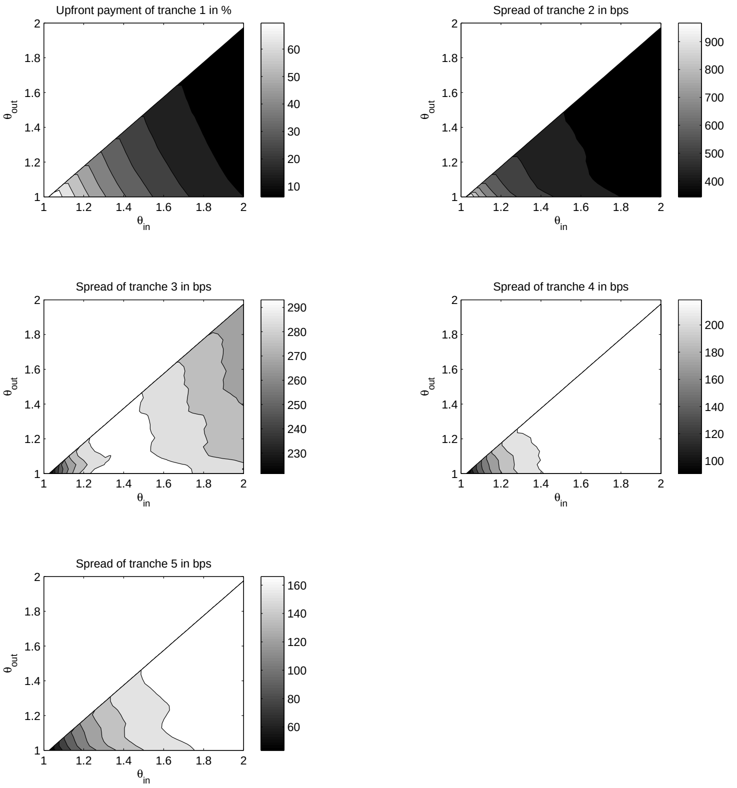


Figure 2: Upfront payment and tranche spreads for the Gumbel model in dependence of $\theta = (\theta_{out}, \theta_{in})$ (2008-05-02).

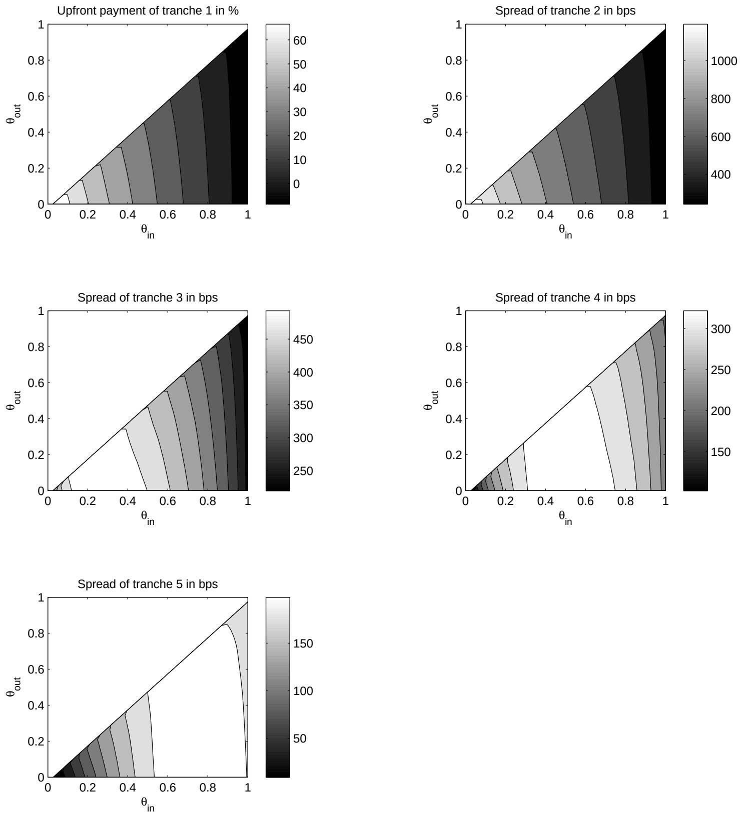
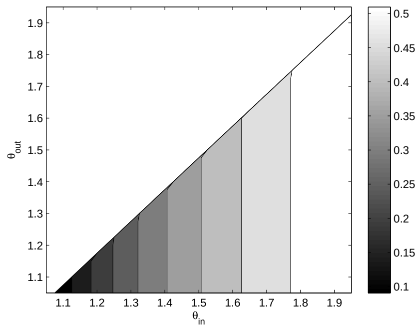
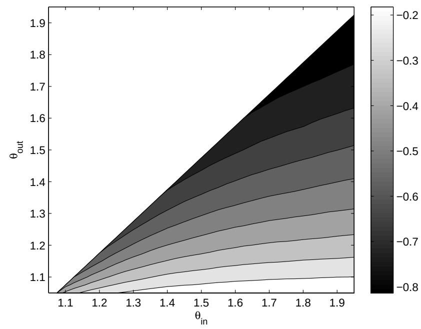


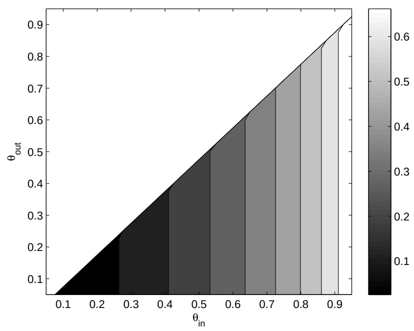
Figure 3: Upfront payment and tranche spreads for the Gauss model in dependence of $\theta = (\theta_{out}, \theta_{in})$ (2008-05-02).



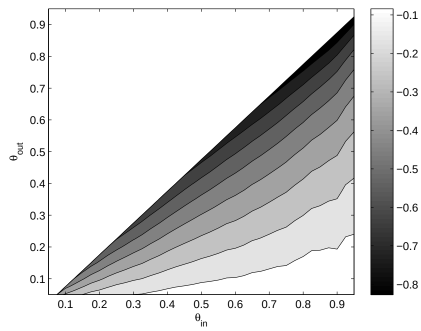
(a) $\rho(5)$ Gumbel.



(b) $\widehat{\rho}^{D,R}(5)$ Gumbel.



(c) $\rho(5)$ Gauss.



(d) $\widehat{\rho}^{D,R}(5)$ Gauss.

Figure 4: Default correlation $\rho(5)$ and average correlation between default rates and recovery rates $\widehat{\rho}^{D,R}(5)$ in the models with Gumbel and Gauss copula (2008-05-02).

to the calibration at 2008-05-02. With an increase of the parameter a the parameter b increases as well to ensure the constant loss given default expectation. Furthermore, the distribution changes from U-shaped to J-shaped to unimodal and finally to a point mass in 0.6 which corresponds to the case of constant recovery. As expected the spreads of the lower tranches increase with a decreasing recovery-rate variability while the spreads of the higher tranches decrease. For tranches 2 - 4 the behaviour of the Gaussian and the Gumbel model is very similar, i.e. the tranche spreads are almost parallel. Only for the first and the fifth tranche there are significant differences. For the Gumbel model the first tranche reacts more sensitive to changes in the shape of the recovery-rate distribution. In contrast, the fifth tranche in the Gaussian copula model is subject to larger changes if the parameter of the recovery distribution varies.

In addition to the sensitivities of the tranche spreads to changes in the model parameters, one thing that is especially of interest for investors in a tranche is the impact of changes in the spread level of the portfolio CDS on the spread levels of each tranche. Buying protection in the portfolio CDS such that the position in a certain tranche of a CDO is insensitive to changes in the level of the portfolio-CDS spread is called delta hedging. To be more precise, the delta of a tranche with respect to the portfolio CDS is defined as the ratio of the change in the spread of the respective tranche to that of the portfolio CDS (see e.g. Masol and Schoutens (2008)). Typically a shift of 1 bp is applied to the portfolio-CDS spread. Assuming constant discount factors, this shift in the portfolio-CDS spread leads to a change in the default intensity, which influences the values of the different CDO tranches. Figure 6 shows the deltas for all tranches in the Gauss and Gumbel model at 2008-05-02. The results for all other considered dates are very similar and therefore omitted here.

It can be easily seen that the hedge ratios remain almost the same regardless of a deterministic or stochastic modelling of recovery rates. In comparison to the Gaussian model, the Gumbel model features higher deltas for the first tranche and lower deltas for all other tranches. On average, the delta of the first tranche is 25% higher in the Gumbel model while the delta of tranche 2 is 20% lower. For tranches 3 - 5 the Gumbel model exhibits deltas that are on average 35% lower than in the Gaussian model.

Another hedging approach for CDO tranches is to hedge the first with the second tranche, which is also sometimes called mezzanine-equity hedging. The corresponding hedge ratio between the two tranches is simply defined as the ratio of the delta of tranche 1 to the delta of tranche 2, both with respect to the portfolio CDS. On average the hedge ratios in the Gumbel model are 60% higher than those in the Gaussian model. According to the Gumbel model for both deterministic and stochastic recoveries the notional of the second tranche has to be 3 - 4 times higher than the notional of the first tranche. In the Gaussian model the notional of the second tranche

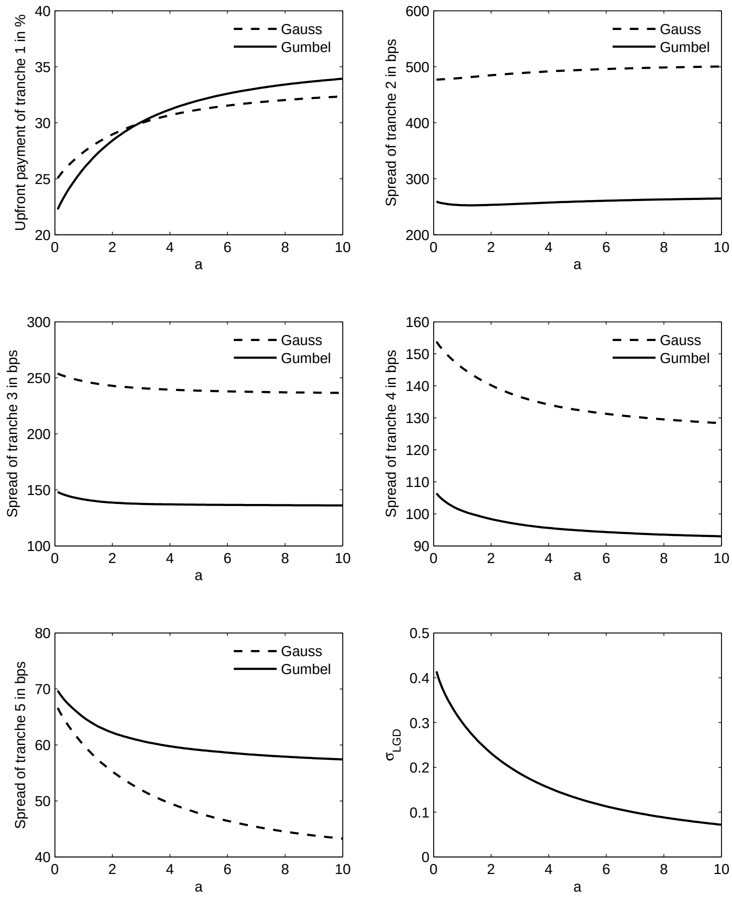


Figure 5: Upfront payment and tranche spreads for Gauss and Gumbel model and σ_{LGD} in dependence of a (2008-05-02).

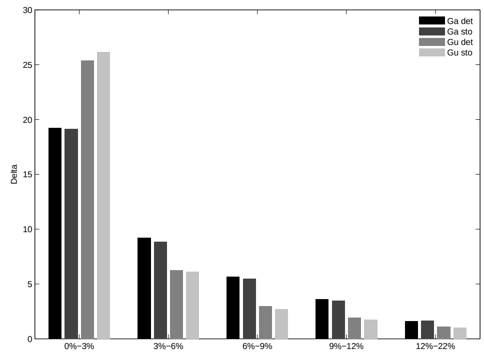


Figure 6: Delta w.r.t. portfolio CDS for different tranches in Gauss and Gumbel model (2008-05-02).

only has to be 1.5 - 2 times the notional of the equity tranche. All the delta hedging results presented here are in line with the results of Masol and Schoutens (2008) who compare hedge ratios of different one-factor Lévy models to those of a Gaussian model.

7 Base Correlation

Similar to the equity market, where it has become standard to quantify the prices of equity options in terms of implied volatility, it has become standard to quantify the spreads of CDO tranches in terms of implied correlation, especially in terms of base correlation as introduced by O'Kane and Livesey (2004). In contrast to the concept of compound correlation, where for each tranche the correlation is chosen such that market and model spread coincide, the concept of base correlation decomposes each tranche into combinations of base tranches, i.e. tranches without subordination covering some interval $[0, u]$. Using the observation that being long a tranche $[l, u]$ coincides with a long position in $[0, u]$ and a short position in $[0, l]$, the base correlation of tranche j, $j = 2, \ldots, J$, can be calculated in an recursive algorithm from the previously computed base correlations and the market spread of tranche j. The base correlation of the equity trance ($j = 1$) coincides with the compound correlation of this tranche. Though being less intuitive, base correlations are more frequently used in practice than compound correlations. One reason for this is the fact that each base correlation only depends on its detachment point. This facilitates the pricing of non-standard tranches via interpolation. Since the Gaussian base correlation introduced in O'Kane and Livesey (2004) has some drawbacks (correlation skew, sensitivity to interpolation scheme), the concept of base correlation has been extended to other models not relying on Gaussian distributions (see e.g. Hooda (2006)

or Garcia *et al.* (2007)).
To define a base correlation[3] curve in the presented modelling framework, i.e. $\theta_{in}(u_j)$, $j = 1, \ldots, 5$, the outer copula parameter θ_{out} is chosen to coincide with the estimate from the global calibration in Section 6. Alternatively, θ_{out} could also be set to the same fixed value for all days and time horizons, but since $\theta_{out} \leq \theta_{in}$ has to be claimed this would imply a rather low level for θ_{out} and hence a low level of correlation between default rates and recovery rates. Then, one can proceed similar to the Gaussian base correlation case as in O'Kane and Livesey (2004) and subsequently bootstrap the default correlation parameter for each base tranche. In the case of deterministic recovery rates, one can start directly with the bootstrapping.
Table 4 and Figure 7 show the base correlation curves in the Gaussian and Gumbel copula model both with and without stochastic recoveries at 02/05/2008 (the results for the other days can be found in Appendix B).

Table 4: Base correlations $\theta_{in}(u_j)$ for 2008-05-02.

	3%	6%	9%	12%	22%
Ga det	0.34	0.46	0.54	0.59	0.73
Ga sto	0.28	0.41	0.49	0.54	0.68
Gu det	1.26	1.27	1.29	1.30	1.40
Gu sto	1.19	1.19	1.19	1.20	1.23

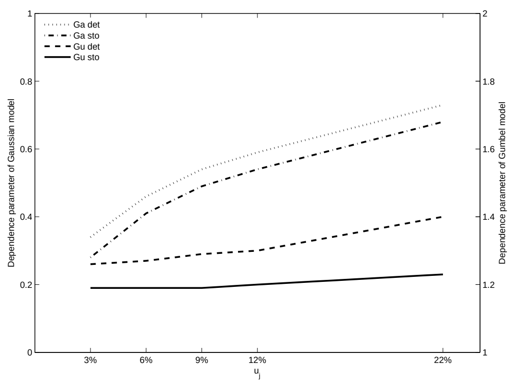


Figure 7: Base correlations $\theta_{in}(u_j)$ for different models with deterministic and stochastic recoveries (2008-05-02).

While the difference in the base correlation curves for the Gaussian model is an almost parallel downward shift, the Gumbel base correlation curve with

[3]Although θ is in the case of Archimedean copulas not the correlation coefficient, the term base correlation is used here for the sake of simplicity when describing a curve of dependence parameters driving the default correlation for different detachment points.

stochastic recovery rates is not only significantly lower but also significantly flatter than the curve in the case of deterministic recoveries. As said above, this property is especially useful for the pricing of non-standard tranches using an inter- or extrapolation scheme of the base correlation curve.

8 Conclusion

In this article, a framework for the joint modelling of default and recovery risk in a portfolio of credit risky assets was presented, which especially accounts for the correlation of defaults on the one hand and correlation of default rates and recovery rates on the other hand. Nested Archimedean copulas were used to model different dependence structures for default correlations and the correlation of default rates and recovery rates. The Kumaraswamy distribution, a very flexible continuous distribution with bounded support, was chosen for the recovery rates to allow for an efficient sampling of the loss process. This is especially important, as in most cases the loss-process distribution will not be given in closed form. Due to the relaxation of the constant 40% recovery assumption and the negative correlation of default rates and recovery rates, this modelling framework is especially suited for distressed market situations and the pricing of super senior tranches. In a numerical example the calibration to CDO tranche spreads of the European iTraxx portfolio was performed to demonstrate the fitting capability of the model. Already in a very simplistic setting of the model, the introduction of stochastic recovery rates consistently decreases the pricing errors compared to the case of deterministic recovery rates. The best calibration results were achieved when the dependences are modelled by a Gumbel copula. In an extension to the Gaussian base correlation framework, significantly flatter base correlation curves could be obtained by using a Gumbel copula and stochastic recovery rates, which simplifies the pricing of non-standard tranches.

References

Altman, E. I. and Kishore, V. M. (1996). Almost everything you wanted to know about recoveries on defaulted bonds. *Financial Analysts Journal*, **52**(6), 57–64.

Amraoui, S. and Hitier, S. (2008). Optimal stochastic recovery rate for base correlation. Working Paper.

Andersen, L. and Sidenius, J. (2004). Extensions to the Gaussian copula: Random recovery and random factor loadings. *Journal of Credit Risk*, **1**(1), 29–70.

Bielecki, T. and Rutkowski, M. (2004). *Credit Risk: Modeling, Valuation and Hedging*. Springer Finance. Springer, Berlin, 2nd edition.

Cherubini, U., Luciano, E., and Vecchiato, W. (2004). *Copula Methods in Finance*. John Wiley & Sons Inc., Chichester, West Sussex.

Ech-Chatbi, C. (2008). CDS and CDO pricing with stochastic recovery. Working Paper.

Emery, K., Cantor, R., and Arner, R. (2004). Recovery rates on North American syndicated bank loans, 1989-2003. Technical report, Moody's Investor Service Global Credit Research.

Feller, W. (1971). *An Introduction to Probability Theory and Its Applications: Volume II*. John Wiley & Sons Inc.

Garcia, J., Goossens, S., Masol, V., and Schoutens, W. (2007). Lévy base correlation. Section of Statistics Technical Report 07-06, K.U. Leuven.

Gupton, G. M. and Stein, R. M. (2005). LossCalc v2: Dynamic Prediction of LGD. Technical report, Moody's Investor Service.

Hering, C., Hofert, M., Mai, J.-F., and Scherer, M. (2009). Constructing nested archimedean copulas with Lévy subordinators. To appear in Journal of Multivariate Analysis.

Hofert, M. (2008). Sampling Archimedean copulas. *Computational Statistics and Data Analysis*, **52**(12), 5163–5174.

Hofert, M. and Scherer, M. (2009). CDO pricing with nested Archimedean copulas. To appear in Quantitative Finance.

Hooda, S. (2006). Explaining base correlation skew using NG (normal-gamma) process. Working Paper.

Joe, H. (1997). *Multivariate Models and Dependence Concepts*. Chapman & Hall, New York.

Johnson, N. L., Kotz, S., and Balakrishnan, N. (1995). *Continuous Univariate Distributions*, volume 2. John Wiley & Sons Inc., New York, 2nd edition.

Kimberling, C. H. (1974). A probabilistic interpretation of complete monotonicity. *Aequationes Mathematicae*, **10**(2-3), 152–164.

Kotz, S. and van Dorp, J. R. (2004). *Beyond Beta. Other continuous families of distributions with bounded support and applications*. World Scientific Publishing Co. Pte. Ltd., Singapore.

Krekel, M. (2008). Pricing distressed CDOs with base correlation and stochastic recovery. Working Paper.

Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. *Journal of Hydrology*, **46**(1-2), 79–88.

Li, D. X. (2000). On default correlation: A copula function approach. *The Journal of Fixed Income*, **9**(4), 43–54.

Marshall, A. W. and Olkin, I. (1988). Families of multivariate distributions. *Journal of the American Statistical Association*, **83**(403), 834–841.

Masol, V. and Schoutens, W. (2008). Comparing some alternative Lévy base correlation models for pricing and hedging CDO tranches. Section of Statistics Technical Report 08-01, K.U. Leuven.

McDonald, J. B. (1984). Some generalized functions for the size distribution of income. *Econometrica*, **52**(3), 647–663.

McNeil, A. J. (2008). Sampling nested Archimedean copulas. *Journal of Statistical Computation and Simulation*, **78**(6), 567–581.

Nelsen, R. B. (1998). *An Introduction to Copulas*. Springer, Berlin, 1st edition.

O'Kane, D. and Livesey, M. (2004). Base correlation explained. Quantitative Credit Research 2004-Q3/4, Lehman Brothers.

Savu, C. and Trede, M. (2006). Hierarchical Archimedean copulas. In *International Conference on High Frequency Finance, Konstanz, Germany, May 2006*.

Schönbucher, P. J. (2003). *Credit Derivatives Pricing Models: Models, Pricing and Implementation*. Wiley Finance Series. John Wiley & Sons Inc., 1st edition.

Schönbucher, P. J. and Schubert, D. (2001). Copula-dependent default risk in intensity models. Working Paper.

Schuermann, T. (2004). What do we know about loss given default? In D. Shimko, editor, *Credit Risk Models and Management 2nd Edition*, chapter 9, pages 249–274. Risk Books, London, 2nd edition.

Vasicek, O. A. (1991). Limiting loan loss probability distribution. Technical report, KMV Corporation.

A Detailed Calibration Results

In this section, the calibration results for all five trading days (2008-02-22, 2008-03-31, 2008-05-02, 2008-06-27, and 2008-07-25) on which the model was tested are presented. Table 5 contains the calibration results for the portfolio-CDS model, Table 6 the calibration errors, Table 7 the calibrated parameters, and Table 8 the default correlations and average correlations between default rates and recovery rates. Finally, Table 9 contains the asymptotic two-sided 95% confidence intervals for the CDO upfront payment and spreads for the different models fitted to 2008-05-02 based on 100,000 Monte Carlo runs.

Table 5: Portfolio-CDS calibration.

Date	$s^{pCDS,market}$	λ	$\bar{p}(1)$	$\bar{p}(5)$
2008-02-22	124.56	0.0207	2.05%	9.85%
2008-03-31	122.29	0.0204	2.02%	9.68%
2008-05-02	63.74	0.0106	1.06%	5.17%
2008-06-27	106.52	0.0178	1.76%	8.49%
2008-07-25	91.64	0.0153	1.52%	7.35%

Table 6: Absolute and relative calibration errors.

		Ga		Gu		opC	
		D_2	D_2^{rel}	D_2	D_2^{rel}	D_2	D_2^{rel}
2008-02-22	det	668.96	53.80%	133.51	10.74%	121.24	9.75%
	sto	590.83	47.52%	70.92	5.70%	80.61	6.48%
2008-03-31	det	972.54	86.87%	277.01	24.74%	295.50	26.40%
	sto	903.76	80.73%	189.66	16.94%	191.93	17.14%
2008-05-02	det	411.87	77.67%	68.15	12.85%	80.26	15.13%
	sto	390.86	73.70%	37.18	7.01%	48.43	9.13%
2008-06-27	det	851.59	94.04%	277.37	30.63%	285.39	31.51%
	sto	788.92	87.12%	197.14	21.77%	218.71	24.15%
2008-07-25	det	676.09	80.96%	157.05	18.81%	165.26	19.79%
	sto	624.57	74.80%	86.46	10.35%	124.17	14.87%

B Detailed Base Correlation Results

In Tables 10 - 13 the base correlation curves at 2008-02-22, 2008-03-31, 2008-06-27, and 2008-07-25 are presented. Similar to Section 7 the difference in the base correlation curves for the Gaussian model with deterministic and stochastic recovery rates is an almost parallel downward shift. For the Gumbel model the base correlations in the case with stochastic recovery

Table 7: Calibrated copula parameters.

		Ga		Gu		opC	
		θ_{in}	θ_{out}	θ_{in}	θ_{out}	θ_{in}	θ_{out}
2008-02-22	det	0.62		1.75		1.79	
	sto	0.52	0.54	1.09	1.68	1.49	1.50
2008-03-31	det	0.46		1.49		1.47	
	sto	0.38	0.38	1.17	1.38	1.30	1.30
2008-05-02	det	0.34		1.26		1.25	
	sto	0.24	0.28	1.11	1.19	1.16	1.16
2008-06-27	det	0.50		1.52		1.50	
	sto	0.41	0.42	1.16	1.40	1.33	1.34
2008-07-25	det	0.45		1.42		1.41	
	sto	0.36	0.37	1.17	1.31	1.27	1.28

Table 8: Default correlation and average correlation between default rates and recovery rates.

		Ga		Gu		opC	
		$\rho(5)$	$\widehat{\rho}^{D,R}(5)$	$\rho(5)$	$\widehat{\rho}^{D,R}(5)$	$\rho(5)$	$\widehat{\rho}^{D,R}(5)$
2008-02-22	det	33.91%	-	50.25%	-	51.82%	-
	sto	27.65%	-68.87%	47.49%	-14.90%	40.25%	-72.54%
2008-03-31	det	22.18%	-	39.75%	-	39.26%	-
	sto	17.00%	-68.38%	33.80%	-36.12%	29.43%	-67.82%
2008-05-02	det	11.85%	-	26.06%	-	25.77%	-
	sto	9.06%	-43.35%	20.68%	-29.03%	18.49%	-48.39%
2008-06-27	det	23.75%	-	40.08%	-	39.74%	-
	sto	18.39%	-65.73%	35.18%	-32.13%	31.81%	-65.20%
2008-07-25	det	19.56%	-	36.12%	-	35.81%	-
	sto	14.75%	-61.55%	29.49%	-37.84%	27.94%	-60.33%

Table 9: Lower and upper boundaries of asymptotic two-sided 95% confidence intervals for the upfront payment and tranche spreads based on 100,000 Monte Carlo runs (2008-05-02).

Model		0-3%	3-6%	6-9%	9-12%	12-22%
Ga det	lower	29.27%	489.46	245.50	138.10	51.10
	upper	29.91%	503.50	255.49	146.08	55.10
Ga sto	lower	29.36%	481.43	237.92	1375.24	50.91
	upper	30.00%	495.41	246.87	1383.24	54.91
Gu det	lower	29.38%	273.80	146.15	101.52	62.02
	upper	29.89%	284.74	154.16	107.51	66.99
Gu sto	lower	29.35%	251.65	134.63	94.63	58.30
	upper	29.85%	261.64	141.64	101.61	63.21
opC det	lower	29.43%	279.15	148.86	104.70	61.87
	upper	29.96%	290.12	156.88	110.71	66.84
opC sto	lower	29.40%	265.51	139.32	93.76	57.97
	upper	29.92%	275.49	146.36	100.76	62.01

rates are significantly smaller and flatter than in the case with deterministic recovery rates.

Table 10: Base correlation $\theta_{in}(u_j)$ for 2008-02-22.

	3%	6%	9%	12%	22%
Ga det	0.62	0.71	0.75	0.78	0.86
Ga sto	0.54	0.64	0.68	0.72	0.81
Gu det	1.75	1.80	1.82	1.84	2.06
Gu sto	1.68	1.72	1.72	1.73	1.88

Table 11: Base correlation $\theta_{in}(u_j)$ for 2008-03-31.

	3%	6%	9%	12%	22%
Ga det	0.46	0.59	0.66	0.71	0.84
Ga sto	0.38	0.52	0.59	0.65	0.80
Gu det	1.49	1.55	1.58	1.62	1.94
Gu sto	1.38	1.42	1.44	1.47	1.66

Table 12: Base correlation $\theta_{in}(u_j)$ for 2008-06-27.

	3%	6%	9%	12%	22%
Ga det	0.50	0.62	0.69	0.74	0.87
Ga sto	0.42	0.55	0.63	0.69	0.83
Gu det	1.52	1.57	1.62	1.67	2.08
Gu sto	1.40	1.44	1.46	1.50	1.78

Table 13: Base correlation $\theta_{in}(u_j)$ for 2008-07-25.

	3%	6%	9%	12%	22%
Ga det	0.45	0.56	0.63	0.68	0.81
Ga sto	0.37	0.49	0.56	0.62	0.77
Gu det	1.42	1.45	1.47	1.50	1.72
Gu sto	1.31	1.32	1.33	1.34	1.46