A general Ornstein-Uhlenbeck stochastic volatility model with Lévy jumps

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Abstract

We present a general class of stochastic volatility models with jumps where the stochastic variance process follows a Lévy-driven Ornstein-Uhlenbeck (OU) process and the jumps in the log-price process follow a Lévy process. This financial market model is a true extension of the Barndorff-Nielsen–Shephard (BNS) model class and can establish a weak link between log-price jumps and volatility jumps. Furthermore, we investigate the weak-link Γ-OU-BNS model as a special case, where we calculate the characteristic function of the logarithmic price in closed form. Moreover, we show that the classical Γ-OU-BNS model can be obtained as a limit of weak-link Γ-OU-BNS models in the Skorokhod topology.

Keywords

financial market model, Barndorff-Nielsen–Shephard model, stochastic volatility, jump-diffusion model, time change, characteristic function, Lévy processes
1 Introduction

Modeling financial assets is a task having craved for more and more sophistication during the last decades. Since complex products increasingly get into focus (i.e. derivatives on realized volatility or variance), more and more stylized facts of time series of asset prices are supposed to be captured. This naturally results into more sophisticated, but also more complex models. Seeding in the groundbreaking works of Samuelson [1965] and Black and Scholes [1973], where the asset price follows a geometric Brownian motion, many extensions and variants have been proposed as, e.g., introducing jumps in the asset price as in Merton [1976]; Kou [2002], accounting for diffusion-style stochastic volatility (e.g. Stein and Stein [1991]; Heston [1993]), combining both approaches (e.g. Bates [1996]), jump-style stochastic volatility models (e.g. Barndorff-Nielsen and Shephard [2001]), general pricing approaches as affine models (e.g. Duffie et al. [2000]). On the other hand, one would like the model to remain interpretable, i.e. being capable of figuring out which parameter accounts for which change in the asset price dynamics.

The model class of Barndorff-Nielsen and Shephard (BNS), which was introduced in Barndorff-Nielsen and Shephard [2001] and extended by Nicolato and Venardos [2003] to account for leverage effects, imposes a Lévy subordinator driven Ornstein-Uhlenbeck structure for the squared volatility process. Furthermore, in the extended notion according to Nicolato and Venardos [2003], upward jumps in the squared volatility process are always accompanied by downward jumps in the asset price. On the other hand, there is empirical evidence (e.g. Jacod and Todorov [2010]) that asset prices and volatility do not always jump together, but there are separate jumps in both processes, which cannot be captured by the BNS model class.

In this paper, we extend the BNS model class in a generic way, accounting for jumps in the asset price as well as the squared volatility process that not necessarily jump together. Therefore, we employ a two-dimensional Lévy process to account for the jumps in the squared volatility process and the asset price process, where the coordinate processes can have any possible dependence structure. The classical BNS model (cf. Barndorff-Nielsen and Shephard [2001]) and the leverage effect BNS model (cf. Nicolato and Venardos [2003]) as well as the two-sided BNS model which can be used for modeling FX rates (cf. Bannör and Scherer [2013]) occur as special cases of this model framework. We discuss different linking mechanisms to introduce dependence between asset price jumps and squared volatility jumps and compute the characteristic function in a semi-closed form, for the general case as well as for special dependence structures. As a special case, we introduce the weak-link Γ-OU-BNS model, where the jumps are driven by coupled compound Poisson processes employing the time change construction presented in Mai et al. [2014]. For this model, we compute the closed-form characteristic function and show that the Γ-OU-BNS model can be regarded as a Skorokhod limit of the weak-link Γ-OU-BNS model.

The remaining paper is organized as follows: In Section 2, we provide a short review of the BNS model class and introduce our extension afterwards. Section 3 deals with the
different possibilities of introducing dependence between Lévy processes. Afterwards, we compute the characteristic function in Section 4 and analyze the limit behavior of the weak-link $\Gamma$-OU-BNS model.

2 A general OU-driven stochastic volatility model class

In this section, we discuss shortcomings of the classical BNS model class, motivating the construction of a general stochastic volatility model class with

- a stochastic variance process following a Lévy-subordinator-driven Ornstein-Uhlenbeck process,
- Lévy jumps in the log-price process, and
- flexible dependence between log-price process jumps and volatility process jumps.

2.1 A short review of the BNS model class

In the seminal paper Barndorff-Nielsen and Shephard [2001], a tractable stochastic volatility model class was presented. In the BNS model class as described in Nicolato and Venardos [2003], the dynamics of the log-price $X = (X_t)_{t \geq 0}$ of a certain asset are governed by the SDEs

\begin{align}
\frac{dX_t}{\sigma_t} &= (\mu + \beta \sigma_t^2) \, dt + \sigma_t \, dW_t + \rho \, dZ_t, \\
\frac{d\sigma_t^2}{\sigma_t} &= -\lambda \sigma_t^2 \, dt + dZ_t,
\end{align}

where $W = (W_t)_{t \geq 0}$ is a Brownian motion, $Z = (Z_t)_{t \geq 0}$ a Lévy subordinator (independent of $W$), $\mu, \beta \in \mathbb{R}$, $\rho \leq 0$, and $\sigma_0^2, \lambda > 0$. The BNS model incorporates stochastic volatility by a $Z$-driven Ornstein-Uhlenbeck process, where volatility suddenly jumps up by a jump in $Z$ and calms down afterwards with exponential decay.\footnote{In many formulations of BNS-type models, an additional time change $t \mapsto \lambda t$ is employed to the process $(Z_t)_{t \geq 0}$, which is mainly for mathematical beauty. From a modeling point of view, the formulation without time change is equivalent.} Furthermore, upward jumps are always accompanied by downward jumps in the asset price process\footnote{In the original paper Barndorff-Nielsen and Shephard [2001], the jump component of the asset price process was nonexistent, which can be materialized by setting $\rho = 0$. The leverage effect was employed by Nicolato and Venardos [2003].}, which accounts for modeling the leverage effect.

This model was, e.g., extended by Bannör and Scherer [2013] to incorporate two-sided jumps in the asset price process to make it more suitable for FX rates modeling. But even in case of equity modeling, the BNS model as described in (1) incorporates the leverage effect in a very strict manner. Every jump in the volatility process is accompanied by a jump in the stock price and vice versa. Obviously, this strong link can seriously be doubted. On the one hand, a sudden jump in the stock price typically triggers...
2.2 The OU-stochastic volatility model with jumps

To mitigate the strong link between asset price jumps and volatility jumps that is postulated in the BNS dynamics in (1), we therefore extend the BNS model by introducing two separate processes as jump drivers in the volatility and asset price process.

**Definition 2.1**

We say that a positive asset price process \( S = (S_t)_{t \geq 0} \) follows an OU-stochastic volatility model with Lévy jumps (OUSVLJ model), if the log-price process \( X = (X_t)_{t \geq 0} = (\log S_t)_{t \geq 0} \) has the dynamics

\[
\begin{align*}
\text{d}X_t &= (\mu + \beta \sigma_t^2) \text{d}t + \sigma_t \text{d}W_t + \text{d}Y_t, \\
\text{d}\sigma_t^2 &= -\lambda \sigma_t^2 \text{d}t + \text{d}Z_t,
\end{align*}
\]  

(2)

where \( W = (W_t)_{t \geq 0} \) is a Brownian motion, \((Y, Z) = (Y_t, Z_t)_{t \geq 0} \) is a 2-dimensional Lévy process such that \( Z \) is a subordinator, \( \mu, \beta \in \mathbb{R} \), and \( \sigma_0^2, \lambda > 0 \).

Different from the classical BNS dynamics in (1), the jumps of the asset price process originate from a separate Lévy process \( Y \), which is possibly dependent of the Lévy subordinator \( Z \) driving the jumps in the volatility process. Some suggestions how to construct dependent Lévy processes are discussed in Section 3. In the remainder of this section, we state some special cases of the introduced model class. First, we express the “classical” BNS model in the light of Definition 2.1.

**Example 2.2 (BNS model as a special case of Definition 2.1)**

By choosing \( Y \) such that \( Y_t := \rho Z_t, \) for all \( t \geq 0, \) and for some \( \rho \leq 0, \) the resulting model with negative linear dependence is exactly the BNS model with dynamics as described in (1).

Furthermore, the construction principle in the model described in Definition 2.1 is very flexible and easily extends several models existing in the literature as, e.g., the jump-diffusion models described in Merton [1976]; Kou [2002]. We briefly summarize special cases of the OUSVLJ model class and their respective construction principles in Table 1.
2.2 The OU-stochastic volatility model with jumps

<table>
<thead>
<tr>
<th>Model</th>
<th>Choice of $Y$</th>
<th>Choice of $Z$</th>
<th>Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Original” BNS</td>
<td>0</td>
<td>$Z$</td>
<td>subordinator $Z$</td>
</tr>
<tr>
<td>“Classical” BNS</td>
<td>$\rho Z$</td>
<td>$Z$</td>
<td>$\rho \leq 0$, subordinator $Z$</td>
</tr>
<tr>
<td>Two-sided BNS</td>
<td>$\rho_+ Z^+ + \rho_- Z^-$</td>
<td>$Z^+ + Z^-$</td>
<td>$\rho_+ \geq 0 \geq \rho_-$, subordinators $Z^+, Z^-$</td>
</tr>
<tr>
<td>OU-SV Merton model</td>
<td>$Y$</td>
<td>$Z$</td>
<td>subordinator $Z$, compound Poisson process $Y$ normal jumps</td>
</tr>
<tr>
<td>OU-SV Kou model</td>
<td>$Y$</td>
<td>$Z$</td>
<td>subordinator $Z$, compound Poisson process $Y$ double exponential jumps</td>
</tr>
<tr>
<td>Weak-link $\Gamma$-OU-BNS</td>
<td>$-Y_T$</td>
<td>$Z_T$</td>
<td>independent subordinators $Y, Z, T$</td>
</tr>
</tbody>
</table>

Table 1 Several special cases of the OUSVLJ model that can either be found in the literature or are simple extensions (OU-SV Merton and Kou model) incorporating Lévy-driven OU stochastic volatility.

A new model fitting in the OUSVLJ class, but not yet existing in the literature, is the weak-link $\Gamma$-OU-BNS model, which employs the time change dependence structure presented in Mai et al. [2014].

**Example 2.3 (Weak-link $\Gamma$-OU-BNS model)**

Let $X = (X_t)_{t \geq 0}$ follow the dynamics

$$dX_t = (\mu + \beta \sigma_t^2) \, dt + \sigma_t \, dW_t - dY_t,$$

$$d\sigma_t^2 = -\lambda \sigma_t^2 \, dt + dZ_t,$$

with $\mu, \beta \in \mathbb{R}$, $\lambda > 0$, $W = (W_t)_{t \geq 0}$ being Brownian motion, and $Y = (Y_t)_{t \geq 0}$, $Z = (Z_t)_{t \geq 0}$ being time-change dependent compound Poisson processes with exponential jump sizes, i.e. there exist independent compound Poisson processes $T = (T_t)_{t \geq 0}$, $U = (U_t)_{t \geq 0}$, $V = (V_t)_{t \geq 0}$ with respective intensities $c_T, c_Y/(c_T - c_Y), c_Z/(c_T - c_Z)$ fulfilling $c_T > 0$, $c_Y, c_Z \in (0, c_T)$ and respective jump size distributions $\text{Exp}(1), \text{Exp}(c_T \eta_Y/(c_T - c_Y)), \text{Exp}(c_T \eta_Z/(c_T - c_Z))$, $\eta_Y, \eta_Z > 0$, such that $Y$ and $Z$ can be represented as the $T$-time-change of the processes $U$ and $V$, i.e. $Y_t := U_{T_t}$ and $Z_t := V_{T_t}$ a.s. for all $t \geq 0$.

The time-change construction method in Example 2.3 yields a tractable and easy-to-simulate structure of dependency. A few properties of this construction are stated in the next remark. For more information on this time-change construction as well as proofs we refer to Mai et al. [2014].

**Remark 2.4 (Properties of the time-change construction in Example 2.3)**

(i) The $T$-subordinated compound Poisson processes $Y := U_T$, $Z := V_T$ are again compound Poisson processes with intensities $c_Y, c_Z$ and jump size distributions $\text{Exp}(\eta_Y), \text{Exp}(\eta_Z)$. 

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(ii) For \( c_{\text{max}} := \max \{ c_Y, c_Z \} \), the correlation coefficient of \((Y_t, Z_t)\) is given by

\[
\text{Corr}[Y_t, Z_t] = \frac{\sqrt{c_Y c_Z}}{c_{\text{max}}}, \quad \text{for all } t \in \mathbb{R},
\]

with \( \kappa = c_{\text{max}} / c_T \in (0, 1) \). We call \( \kappa \) the jump correlation parameter. In particular, correlation coefficients ranging from zero to \( \sqrt{c_Y c_Z} / c_{\text{max}} \) are possible, and the correlation does not depend on the point in time \( t \).

(iii) Due to the joint time change, the compound Poisson processes \( Y \) and \( Z \) are stochastically dependent. Moreover, it can be shown that the dependence structure of the two-dimensional process \((Y, Z)\) is solely driven by the jump correlation parameter \( \kappa \).

Figure 1 shows simulated paths for the classical BNS model and the weak-link \( \Gamma \)-OU-BNS model. The graphs in the first row show typical asset price paths of the two models. Corresponding to these paths, the graphs beneath exhibit the volatility process and the daily log-returns. On the left side, the jump correlation parameter of the weak-link \( \Gamma \)-OU-BNS model is set to be 80%, on the right side, this parameter is 20%. Thus, we have a strong dependence between the asset price jumps and jumps in the volatility on the left side and a weak dependence on the right side. For the sake of good comparability, the Brownian motions and the asset jump processes of both models coincide within one dependence configuration. Therefore, the difference between the two models is determined solely by the jumps in the volatility process. Moreover, the compound Poisson processes driving the volatility are identically distributed. One can easily see that the volatility jumps are uncoupled from the asset price jumps in the weak-link \( \Gamma \)-OU-BNS model, i.e. there exist asset price jumps without simultaneous volatility jumps and, on the other hand, there are volatility rises without negative asset price jumps. The higher the jump dependence correlation parameter in the weak-link \( \Gamma \)-OU-BNS model is, the higher seems the resemblance to the classical BNS model. This impression is confirmed by a mathematical proof in Theorem 4.7.

Generalizing the two-sided BNS model presented in Bannör and Scherer [2013], one can also establish a more complex dependence structure by employing linear dependence from an \( n \)-dimensional Lévy subordinator with independent coordinate processes. Hence, we formulate the linear dependence BNS model by employing matrix-vector notation.

**Example 2.5 (Linear dependence BNS model)**

Let \( Z = (Z_1, \ldots, Z_n)_{t \geq 0} \) be an \( n \)-dimensional Lévy subordinator with independent coordinate processes and let \( \rho \in \mathbb{R}^n \). Furthermore, let \( \xi \in \{0, 1\}^n \) with at least one \( \xi_j = 1 \), \( j = 1, \ldots, n \). Then the model following the dynamics

\[
\begin{align*}
\text{d}X_t &= (\mu + \beta \sigma_t^2) \text{d}t + \sigma_t \text{d}W_t + \rho' \text{d}Z_t, \\
\text{d}\sigma_t^2 &= -\lambda \sigma_t^2 \text{d}t + \xi' \text{d}Z_t,
\end{align*}
\]

is called the general linear dependence BNS model.
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Figure 1 Sample paths of the asset price process, the volatility process, and the daily log-returns for the classical BNS model and the weak-link Γ-OU-BNS model. Left: $\kappa = 80\%$, right: $\kappa = 20\%$. 
Choosing \( n = 2, \xi = (1, 1)' \), and \( \rho = (\rho_-, \rho_+) \) with \( \rho_- \leq 0 \leq \rho_+ \), the linear dependence model reduces to the two-sided BNS model of Bannör and Scherer [2013].

### 3 The dependence structure between volatility and asset price jumps

The main difference between the classical BNS model and the OUSVLJ model lies in the relationship between volatility and asset jumps: In the classical BNS model, every (upward) volatility jump is accompanied by a downward jump in the asset price process, while the parameter \( \rho \) steers the magnitude of the asset price process jump. Conversely, in the general OU-stochastic volatility model, this close relationship is not anymore the case: Similar to the development of the Cox-Ingersoll-Ross-type stochastic volatility models from Heston [1993] over Bates [1996] to Duffie et al. [2000], the dependence of volatility and asset prices becomes more sophisticated, since we only assume to have some dependence structure preserving the two-dimensional Lévy structure of \( Y \) and \( Z \).

In this section, we discuss different possibilities of establishing dependence between Lévy processes.

#### 3.1 Possibilities of introducing dependence between Lévy processes

There are several construction principles to obtain a two-dimensional Lévy process with dependence from two one-dimensional Lévy processes. Deelstra and Petkovic [2010] neatly summarize three possibilities to construct dependent pure-jump Lévy processes:

1. Linear combination of independent Lévy processes
2. Joint time change of independent Lévy processes by a Lévy subordinator
3. Linking the according Lévy measures by a Lévy copula

Obviously, other constructions are possible to create a dependent two-dimensional Lévy process (as, e.g., direct construction from a two-dimensional infinitely divisible law), but the above construction principles provide flexible instruments, where one starts with independent Lévy processes and results with dependent ones. A standard reference about financial modeling with Lévy processes with a focus on univariate Lévy processes is Cont and Tankov [2004].

In most special cases presented in Table 1, one of the construction principles is applied. The classical and two-sided BNS model both stem from linear combinations of independent Lévy subordinators, resulting in a very immediate link between the processes: In these models, jumps in the volatility process are always accompanied by jumps in the asset price process. This property arises directly from introducing dependence via linear dependence between the diffusion components is provided by using correlated Brownian motions.
4 Pricing in the general OU stochastic volatility model

When introducing dependence by joint time change between independent Lévy processes, the link between the jumps in the volatility and asset price process becomes weaker and more blurry. Joint time change of two independent Lévy processes causes the probability of joint jumps to rise due to "common clocking", but does not necessarily imply simultaneous jumps. Hence, when introducing weak links between jumps, joint time change is a tractable method. An example where a weak link between Γ-OU-BNS models is established, involving joint time-change, is discussed in Mai et al. [2014]: For independent compound Poisson processes with exponential jump sizes, a time change with another exponential jump size compound Poisson process preserves the compound Poisson structure. This construction can similarly be used to establish a weak link within a Γ-OU-BNS model, resulting in the weak-link Γ-OU-BNS model from Example 2.3. Here, we do not employ a stochastic time change to model some kind of business time (as, e.g., in Luciano and Schoutens [2006]), but we solely use the time change construction to establish some kind of weak dependence between the Lévy processes that joint jumps occur in a stochastic manner.

Linking the respective Lévy measures by Lévy copulas was promoted by Tankov [2004] and Kallsen and Tankov [2006]. Analogously to linking marginal distributions by a copula (as described in Nelsen [2006]), one may link univariate, independent Lévy measures by a Lévy copula. Lévy copulas are functions fulfilling some regularity conditions linking the tail integrals w.r.t. the Lévy measures. Sklar's theorem for Lévy copulas (cf. [Kallsen and Tankov, 2006, Theorem 3.6]) states that this construction principle is a universal one, i.e. every dependence structure in multidimensional Lévy processes can be constructed from independent Lévy processes, linked by some suitable Lévy copula. For a pure mathematics point of view, the universal concept of Lévy copulas makes the above mentioned constructions redundant. But for pricing purposes, a closed-form characteristic function of (integrated) variance and asset price process is typically helpful. Furthermore, a tractable simulation scheme for Monte Carlo simulation is essential.

With linear combination and joint time change of independent Lévy processes, the characteristic function of the factors can be calculated at least in a semi-closed form and a simulation scheme is immediately provided, while linking independent Lévy processes with Lévy copulas typically exhibits cumbersomeness concerning these issues.

4 Pricing in the general OU stochastic volatility model

To ensure quick and convenient valuation of plain vanilla derivatives (i.e. for calibration purposes), many practitioners rely on Fourier pricing methods like FFT pricing (i.e. Carr and Madan [1999]; Raible [2000]) or the COS method described in Fang and Oosterlee [2008]. Therefore, the knowledge of the characteristic function in a (semi-)closed form is essential. In this section, we compute the characteristic functions of several (sub-)types
of the general OU stochastic volatility model. Furthermore, we investigate the limit behavior of the weak-link $\Gamma$-OU-BNS model introduced in Example 2.3 and obtain the classical BNS model as a Skorokhod limit.

To avoid confusion about the terminology, we define the Laplace exponent of a $d$-dimensional Lévy process $L = (L_t)_{t \geq 0}$, $d \in \mathbb{N}$, by $\psi_L(u) := \log \mathbb{E}[\exp(u'L_1)]$ for $u \in \mathbb{C}^d$ provided the expected value exists.

We start with the computation of the finite-dimensional distribution of the log-price process in Theorem 4.1. This is done by calculating the joint characteristic function of the log-price process at finitely many points in time.

Theorem 4.1 (Finite-dimensional distribution of the log-price process)

Let the logarithmic price process $(X_t)_{t \geq 0}$ follow the SDE according to equation (2). Set $n \in \mathbb{N}$, $0 = t_0 \leq t_1 < \cdots < t_n$, and $u_1, \ldots, u_n \in \mathbb{R}$. Define $\tilde{u}_j := \sum_{k=j}^{n} u_k$ for all $1 \leq j \leq n$. Then,

$$
\mathbb{E} \left[ \exp \left( \sum_{j=1}^{n} i u_j X_{t_j} \right) \right] = \exp \left( i \tilde{u}_1 X_0 + \sum_{j=1}^{n} \left( i \mu \tilde{u}_j t_j + f(\tilde{u}_j) \epsilon(t_{j-1}, t_j) e^{-\lambda t_j - \sigma_0^2} + \int_{t_{j-1}}^{t_j} \psi(y, z) (i \tilde{u}_j, a_j(t)) \, dt \right) \right),
$$

with

$$
f(u) := \frac{1}{\lambda} \left( iu\beta - \frac{u^2}{2} \right),
$$

$$\epsilon(s, t) := 1 - e^{\lambda(s-t)},$$

$$a_j(t) := f(\tilde{u}_j) \epsilon(t, t_j) + \sum_{k=j+1}^{n} f(\tilde{u}_k) \epsilon(t_{k-1}, t_k) e^{\lambda(t-t_k-1)}.$$

Proof

First, we state two simple calculations, which are needed later in the proof.

(i) By the definition of the squared volatility process, we get

$$
\int_{t_{j-1}}^{t_j} \sigma_t^2 \, dt = \int_{t_{j-1}}^{t_j} \frac{1 - e^{\lambda(t-t_j)}}{\lambda} \, dZ_t + \frac{1 - e^{\lambda(t_j-t_{j-1})}}{\lambda} \sigma_{t_{j-1}}^2
$$

$$
= \int_{t_{j-1}}^{t_j} \frac{1 - e^{\lambda(t-t_j)}}{\lambda} \, dZ_t + \frac{1 - e^{\lambda(t_j-t_{j-1})}}{\lambda} \left( e^{-\lambda t_j - \sigma_0^2} + \int_{0}^{t_{j-1}} e^{\lambda(t-t_j-1)} \, dZ_t \right).
$$
(ii) Simple rearranging of summands yields

\[ \sum_{j=1}^{n} f(\tilde{u}_j) \int_{0}^{t_{j-1}} \epsilon(t_{j-1}, t_j) e^{\lambda(t-t_{j-1})} dZ_t = \sum_{j=1}^{n} \sum_{k=1}^{t_{j-1}} f(\tilde{u}_j) e(t_{j-1}, t_j) e^{\lambda(t-t_{j-1})} dZ_t \]

\[ = \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} f(\tilde{u}_k) e(t_{j-1}, t_j) e^{\lambda(t-t_{j-1})} dZ_t. \]

Second, we start calculating the joint characteristic function,

\[ E \left[ \exp \left( \sum_{j=1}^{n} i u_j X_{t_j} \right) \right] = E \left[ \exp \left( i \tilde{u}_1 X_{t_0} + \sum_{j=1}^{n} i \tilde{u}_j (X_{t_j} - X_{t_{j-1}}) \right) \right] \]

\[ = E \left[ \exp \left( i \tilde{u}_1 X_{t_0} + \sum_{j=1}^{n} i \tilde{u}_j \left( \int_{t_{j-1}}^{t_j} (\mu + \beta \sigma_j^2) dt + \int_{t_{j-1}}^{t_j} \sigma_t dW_t + Y_{t_j} - Y_{t_{j-1}} \right) \right) \right]. \]

Conditioning on the trajectories of \( Y \) and \( Z \) yields

\[ = \exp \left( i \tilde{u}_1 X_{t_0} + i \mu \sum_{j=1}^{n} u_j t_j \right) \left[ \exp \left( \sum_{j=1}^{n} \left( i \beta \tilde{u}_j - \frac{1}{2} \tilde{u}_j^2 \right) \int_{t_{j-1}}^{t_j} \sigma_t^2 dt + \sum_{j=1}^{n} i \tilde{u}_j (Y_{t_j} - Y_{t_{j-1}}) \right) \right], \]

and by (i), we obtain

\[ = E \left[ \exp \left( \sum_{j=1}^{n} \left( f(\tilde{u}_j) \left( \int_{t_{j-1}}^{t_j} \epsilon(t, t_j) dZ_t + \epsilon(t_{j-1}, t_j) \int_{0}^{t_{j-1}} e^{\lambda(t-t_{j-1})} dZ_t \right) + i \tilde{u}_j (Y_{t_j} - Y_{t_{j-1}}) \right) \right) \right] \]

\[ \times \exp \left( i \tilde{u}_1 X_{t_0} + \sum_{j=1}^{n} \left( i \mu \tilde{u}_j t_j + f(\tilde{u}_j)e(t_{j-1}, t_j)e^{-\lambda t_{j-1}\sigma_0^2} \right) \right). \]

Using (ii) yields

\[ = E \left[ \exp \left( \sum_{j=1}^{n} \left( \int_{t_{j-1}}^{t_j} f(\tilde{u}_j) \epsilon(t, t_j) + \int_{k=j+1}^{n} f(\tilde{u}_k) e(t_{k-1}, t_k) e^{\lambda(t-t_{k-1})} dZ_t + i \tilde{u}_j (Y_{t_j} - Y_{t_{j-1}}) \right) \right) \right] \]

\[ \times \exp \left( i \tilde{u}_1 X_{t_0} + \sum_{j=1}^{n} \left( i \mu \tilde{u}_j t_j + f(\tilde{u}_j)e(t_{j-1}, t_j)e^{-\lambda t_{j-1}\sigma_0^2} \right) \right). \]
Since the 2-dimensional Lévy process \((Y, Z)\) has independent increments, we get

\[
\prod_{j=1}^{n} \mathbb{E}\left[ \exp \left( \int_{t_{j-1}}^{t_{j}} \left( f(\tilde{u}_j)\epsilon(t, t_{j}) + \sum_{k=j+1}^{n} f(\tilde{u}_k)\epsilon(t_{k-1}, t_{k})e^{\lambda(t_{k-1})} \right) \right) + i\tilde{u}_j (Y_{t_{j}} - Y_{t_{j-1}}) \right] \times \exp \left( i\tilde{u}_1 X_0 + \sum_{j=1}^{n} \left( i\mu_j t_{j} + f(\tilde{u}_j)\epsilon(t_{j-1}, t_{j})e^{-\lambda t_{j-1}} \sigma_0^2 + \int_{t_{j-1}}^{t_{j}} \psi_{(Y, Z)}(i\tilde{u}_j, a_j(t)) \, dt \right) \right). \]

To obtain the final step, we have to apply the moment-generating function for stochastic integrals w.r.t. Lévy integrators, i.e. a straightforward multivariate generalization of Ebberlein and Raible [1999]

\[
\exp \left( i\tilde{u}_1 X_0 + \sum_{j=1}^{n} \left( i\mu_j t_{j} + f(\tilde{u}_j)\epsilon(t_{j-1}, t_{j})e^{-\lambda t_{j-1}} \sigma_0^2 + \int_{t_{j-1}}^{t_{j}} \psi_{(Y, Z)}(i\tilde{u}_j, a_j(t)) \, dt \right) \right). \]

As an immediate corollary, we obtain a semi-closed form for the characteristic function of the logarithmic price.

**Corollary 4.2 (Characteristic function of the logarithmic price process)**

Let the logarithmic price process \((X_t)_{t \geq 0}\) follow the SDE according to equation (2). Define the abbreviations

\[
f(u) := \frac{1}{\lambda} \left( iu\beta - \frac{u^2}{2} \right), \\
\epsilon(s, t) := 1 - e^{\lambda(s-t)}. \]

Then, the characteristic function of the logarithmic price \(X_t, t \geq 0\) is given by

\[
\phi_{X_t}(u) = \exp \left( iuX_0 + iu\mu t + f(u)e(0, t)\sigma_0^2 + \int_{0}^{t} \psi_{(Y, Z)}(iu, f(u)\epsilon(s, t)) \, ds \right), \]

denoting by \(\psi_{(Y, Z)}\) the Laplace exponent of the two-dimensional Lévy process \((Y_t, Z_t)_{t \geq 0}\).

Since the joint Laplace exponent of a two-dimensional Lévy process (which appears in the expressions in Theorem 4.1 and Corollary 4.2) may be a cumbersome object, we calculate it for the special case of dependence arising from joint time change. The corresponding calculations for dependence arising from linear dependence are straightforward, therefore, we omit them here.
Lemma 4.3 (Joint Laplace exponent in case of joint time change)
If the two-dimensional Lévy process \((Y, Z) = (Y_t, Z_t)_{t \geq 0}\) is constructed by jointly time-changing two independent Lévy processes \(U = (U_t)_{t \geq 0}, V = (V_t)_{t \geq 0}\), i.e. it exists a Lévy subordinator \(T = (T_t)_{t \geq 0}\) such that \(Y_t = U_{T_t}\) and \(Z_t = V_{T_t}\) a.s. for all \(t > 0\), then the joint characteristic function of \((Y_t, Z_t)\) can be calculated as
\[
E[\exp(i(uU_{T_t} + vV_{T_t}))] = \exp(t\psi_T(\psi_U(iu) + \psi_V(iv)))
\]
where \(\psi\) denotes the Laplace exponent of the corresponding process.

Proof
The claim follows easily by conditioning on \(T_t\),
\[
E[\exp(i(uU_{T_t} + vV_{T_t}))] = E[\exp((\psi_U(iu) + \psi_V(iv))T_t)]
= \exp(t\psi_T(\psi_U(iu) + \psi_V(iv))) \quad \square
\]

For the weak-link \(\Gamma\)-OU-BNS model, we immediately obtain an explicit expression for the joint Laplace exponent.

Remark 4.4 (Joint Laplace exponent for the weak-link \(\Gamma\)-OU-BNS model)
In the special case of \(U, V\) being compound Poisson processes with exponential jumps and joint time change \(T\) as well being a compound Poisson process with exponential jumps and an average jump height of 1, the joint Laplace exponent reduces to
\[
\psi_{(Y, Z)}(u, v) = \psi_T(\psi_U(u) + \psi_V(v)) = c_T \frac{c_U \frac{u}{\eta_U - u} + c_V \frac{v}{\eta_V - v}}{1 - (c_U \frac{u}{\eta_U - u} + c_V \frac{v}{\eta_V - v})}.
\]

Finally, by plugging in the values for the parameters \(c_U, c_V, c_T, \eta_U\) and \(\eta_V\) (cf. Example 2.3), we get a closed-form expression for the joint Laplace exponent,
\[
\psi_{(Y, Z)}(u, v) = c_T \frac{c_Y u \frac{u}{\eta_U - u} + c_Z v \frac{v}{\eta_V - v}}{1 - (c_T \frac{u}{\eta_U - u} + c_T \frac{v}{\eta_V - v})} = c_T \frac{c_Y u \frac{u}{\eta_U - u} + c_Z v \frac{v}{\eta_V - v}}{1 - (c_T \frac{u}{\eta_U - u} + c_T \frac{v}{\eta_V - v})}
\]
\[
= c_T \frac{c_Y u \frac{u}{\eta_U - u} + c_Z v \frac{v}{\eta_V - v}}{1 - (c_T \frac{u}{\eta_U - u} + c_T \frac{v}{\eta_V - v})} \quad (3)
\]
Moreover, the corresponding integral appearing in Corollary 4.2 can be computed in a closed-form expression. This is done in Theorem 4.6.

As a special case, we can consider a one-sided time-change construction, where the terms become slightly simpler.

Remark 4.5 (Joint Laplace exponent of a one-sided time-change construction)
If the two-dimensional Lévy process \((Y, Z) = (Y_t, U_{Y_t})_{t \geq 0}\) is constructed by two independent compound Poisson processes \(Y, U\) with intensities \(c_Y, c_U\) and jump size distributions \(\text{Exp}(\eta_Y), \text{Exp}(\eta_U)\), then the joint Laplace exponent is given by
\[
\psi_{(Y, Z)}(u, v) = \psi_Y(u + \psi_U(v)) = c_Y \frac{u(\eta_U - v) + c_U v}{\eta_Y - u}(\eta_U - v) - c_U v.
\]
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This construction is slightly simpler, but less flexible than the weak-link construction. In particular, a one-sided time-change construction only allows for separate jumps in one component, while the jumps in the other component always occur jointly. Later, we show that a model resulting from such a time change construction can be obtained as the Skorokhod limit of the weak-link $\Gamma$-OU-BNS model (cf. Theorem 4.7).

Having now gathered all ingredients, we now show that the characteristic function for the weak-link $\Gamma$-OU-BNS model can be calculated in closed form.

**Theorem 4.6 (Characteristic function of the weak-link $\Gamma$-OU-BNS model)**

Let $X = (X_t)_{t \geq 0}$ follow a weak-link $\Gamma$-OU-BNS model (cf. Example 2.3). Then the characteristic function can be calculated in closed form and is given by

$$\log \phi_X(u) = iu(X_0 + \mu t) + f(u)e(0,t)\sigma_0^2 - \frac{cT}{\lambda} (\alpha(u) + \delta(u)) \log (\gamma(u)) + cT\delta(u)t,$$

with the following abbreviations

$$f(u) := \frac{1}{\lambda} \left( iu\beta - \frac{u^2}{2} \right),$$

$$\epsilon(s,t) := 1 - e^{\lambda(s-t)},$$

$$g(u) := c_T \epsilon_Y + iuc_T - c_Z + iuc_T (c_T - c_Y),$$

$$h(u) := c_T^2 \eta_Y + iuc_T^2 - c_Y c_Z,$$

$$k(u) := iuc_T \epsilon_Y \eta_Z,$$

$$l(u) := c_T^2 \eta_Z \eta_Y + iu,$$

$$\alpha(u) := \frac{g(u)}{h(u)},$$

$$\delta(u) := \frac{f(u)g(u) - k(u)}{l(u) - f(u)h(u)},$$

$$\gamma(u) := \frac{l(u)}{l(u) - \epsilon(0,t)f(u)h(u)}.$$

**Proof**

Note first that by Corollary 4.2, the only thing left to show is that

$$\int_0^t \psi_{(-Y,Z)}(iu, f(u)e(s,t)) \, ds = -\frac{cT}{\lambda} (\alpha(u) + \delta(u)) \log(\gamma(u)) + cT\delta(u)t.$$

Plugging in the specific joint Lévy exponent $\psi_{(Y,Z)}$ (cf. Remark 4.4), we obtain

$$\int_0^t \psi_{(-Y,Z)}(iu, f(u)e(s,t)) \, ds = \int_0^t \psi_{(Y,Z)}(-iu, f(u)e(s,t)) \, ds$$

$$= \int_0^t -f(u)g(u) \exp(-\lambda t) \exp(\lambda s) + f(u)g(u) - k(u) \frac{f(u)h(u) \exp(-\lambda t) \exp(\lambda s) - h(u)f(u) + l(u)}{f(u)h(u) \exp(-\lambda t) \exp(\lambda s) - h(u)f(u) + l(u)} \, ds$$
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after applying some algebraic transformations and substitutions and adopting the terminology of the theorem. To solve this integral, we remark that for arbitrary \( x, y, z, w \in \mathbb{C} \) with \( z \exp(\lambda s) + w \neq 0 \) for all \( s \in [0, t] \), the derivative of the function

\[
\zeta(s) := \frac{1}{\lambda} \left( \frac{x}{z} - \frac{y}{w} \right) \log(z \exp(\lambda s) + w) + \frac{y}{w}, \quad s \in [0, t]
\]

is exactly

\[
\zeta'(s) = \frac{x \exp(\lambda s) + y}{z \exp(\lambda s) + w}
\]

for all \( s \in [0, t] \). Hence, defining

\[
x := -f(u)g(u)\exp(-\lambda t), \quad y := f(u)g(u) - k(u)
\]

\[
z := f(u)h(u)\exp(-\lambda t), \quad w := -h(u)f(u) + l(u),
\]

yields

\[
\int_0^t \psi_{(-Y,Z)}(iu, f(u)e(s, t)) \, ds
\]

\[
= c_T(t - \zeta(0)) = c_T \left( \frac{x}{z} - \frac{y}{w} \right) \log \left( \frac{z \exp(\lambda t) + w}{z + w} \right) + c_T \frac{yt}{w},
\]

which results in the proclaimed characteristic function. \( \square \)

As described in Remark 2.4, the dependence structure between the squared volatility process and the asset price process in the weak-link \( \Gamma \)-OU-BNS model can be completely described by the time change intensity \( c_T \), or, alternatively, by the jump correlation parameter \( \kappa = \max\{c_Y, c_Z\}/c_T \), which floats in the open unit interval \((0, 1)\) and may therefore be more interpretable than the time change intensity \( c_T \).

Obviously, the time-change construction in the weak-link \( \Gamma \)-OU-BNS model always establishes nonlinear dependence between the asset price and the squared volatility process. Therefore, the weak-link \( \Gamma \)-OU-BNS model is not a true extension of the classical \( \Gamma \)-OU-BNS model. But we can show that the classical \( \Gamma \)-OU-BNS model occurs as a limit model in the Skorokhod topology as motivated in Figure 1. Thus, the weak-link \( \Gamma \)-OU-BNS model can be considered as an extension where the \( \Gamma \)-OU-BNS model occurs as a limiting case. Regarding the limit \( \kappa \to 1 \), the dependence structure is simplified to linear dependence compared to the time change induced dependence in the weak-link \( \Gamma \)-OU-BNS model. Furthermore, dependent on the setting, a dependence structure resulting from a one-sided time change construction (cf. Remark 4.5) occurs as a limit behavior. Theorem 4.7 investigates the limit behavior of the weak-link \( \Gamma \)-OU-BNS model.

**Theorem 4.7 (Skorokhod limit of the weak-link \( \Gamma \)-OU-BNS model)**

Let the log-price process \( X_\kappa \) be given by a weak-link \( \Gamma \)-OU-BNS model (cf. Example 2.3), with \( \kappa \) being the respective jump correlation parameter as defined in Remark 2.4. Then,
the process $X_\kappa$ converges to a process $X$ for $\kappa \nearrow 1$ in the Skorokhod topology. The structure of the limiting process $X$ depends on the intensities $c_Z$ and $c_Y$ in the following way:

- $c_Y > c_Z$:
  $X$ can be represented by a construction as described in Remark 4.5, i.e., by the two-dimensional Lévy process $(-\tilde{Y}_t, \tilde{Z}_t)_{t \geq 0} = (-\tilde{Y}_t, \tilde{U}_t)_{t \geq 0}$, where $\tilde{Y}, \tilde{U}$ are independent compound Poisson processes with intensities $c_Y, c_Z \eta_Y / (c_Y - c_Z)$ and jump size distribution $\text{Exp}(\eta_Y), \text{Exp}(c_Y \eta_Z / (c_Y - c_Z))$.

- $c_Z > c_Y$:
  $X$ is given by a construction as described in Remark 4.5, i.e., by the two-dimensional Lévy process $(-\tilde{Y}_t, \tilde{Z}_t)_{t \geq 0} = (-\tilde{U}_t, \tilde{Z}_t)_{t \geq 0}$, where $\tilde{Z}, \tilde{U}$ are independent compound Poisson processes with intensities $c_Z, c_Y \eta_Z / (c_Z - c_Y)$ and jump size distribution $\text{Exp}(\eta_Z), \text{Exp}(c_Z \eta_Y / (c_Z - c_Y))$.

- $c_Y = c_Z$:
  $X$ is given by a classical BNS model, as described in Section 2.1, i.e., by the two-dimensional Lévy process $(\rho \tilde{Z}_t, \tilde{Z}_t)_{t \geq 0}$, where $\rho = -\eta_Y / \eta_Z$ and $\tilde{Z}$ is a compound Poisson process with intensity $c_Z$ and jump size distribution $\text{Exp}(\eta_Z)$.

**Proof**

First, we note that the distribution of the semimartingale characteristics of $X_\kappa$ does not depend on the jump correlation parameter $\kappa$. Therefore, by [Jacod and Shiryaev, 2003, Theorem IV.4.18, Theorem IV.3.18], it suffices to show that the finite-dimensional distribution of $X_\kappa$ converges to the finite-dimensional distribution of $X$. Using Theorem 4.1, the problem can be reduced to show that the Laplace exponent of $(\tilde{Y}_\kappa, \tilde{Z}_\kappa)$ converges pointwise to the Laplace exponent of $(\tilde{Y}, \tilde{Z})$.

Consider the case, where $c_Y > c_Z$, then by Remark 4.4,

$$\lim_{\kappa \nearrow 1} \psi_{(\tilde{Y}_\kappa, \tilde{Z}_\kappa)}(u, v) = c_Y \frac{u}{\eta_Y} + c_Z \frac{v}{c_Y \eta_Z - v(c_Y - c_Z)} = c_Y \frac{u}{\eta_Y} + c_Z \frac{v}{c_Y \eta_Z - v(c_Y - c_Z)}$$

$$\left(1 - \frac{u}{\eta_Y} \frac{c_Z v}{c_Y \eta_Z - v(c_Y - c_Z)}\right) = c_Y \frac{u}{\eta_Y} + c_Z \frac{v}{c_Y \eta_Z - v(c_Y - c_Z)}$$

which is the claimed formula as in Remark 4.5. In case of $c_Y < c_Z$, we get the result analogously.

Now assume $c_Y = c_Z$, then

$$\lim_{\kappa \nearrow 1} \psi_{(\tilde{Y}_\kappa, \tilde{Z}_\kappa)}(u, v) = c_Z \frac{u + v \eta_Z}{\eta_Y - u - v \eta_Z},$$

which coincides with the joint characteristic function of $\tilde{Z}$ and $\rho \tilde{Z}$. $\Box$

## 5 Conclusion

In this paper, we have extended the jump-diffusion type Barndorff-Nielsen–Shephard model class of Barndorff-Nielsen and Shephard [2001] to the more general OUSVLJ model
References

class, where the squared volatility structure follows a Lévy subordinator driven Ornstein–
Uhlenbeck process, but the strong link between the jumps in the squared volatility
process and the asset price process is ameliorated by introducing weaker dependence
structures, capturing more realistic behavior of the asset price process. As a tractable
member, we have introduced the weak-link $\Gamma$-OU-BNS model, relying on a time change
construction. We have shown that the weak-link $\Gamma$-OU-BNS model has a closed-form
characteristic function and the classical $\Gamma$-OU-BNS model results as a limit model.

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